

# ALGEBRAIC $K$ -THEORY OF THE FIRST MORAVA $K$ -THEORY

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**ABSTRACT.** For a prime  $p \geq 5$ , we compute the algebraic  $K$ -theory modulo  $p$  and  $v_1$  of the mod  $p$  Adams summand, using topological cyclic homology. On the way, we evaluate its modulo  $p$  and  $v_1$  topological Hochschild homology. Using a localization sequence, we also compute the  $K$ -theory modulo  $p$  and  $v_1$  of the first Morava  $K$ -theory.

## 1. INTRODUCTION

In this paper we continue the investigation from [AR02] and [Aus10] of the algebraic  $K$ -theory of topological  $K$ -theory and related  $S$ -algebras. Let  $\ell_p$  be the  $p$ -complete Adams summand of connective complex  $K$ -theory, and let  $\ell/p = k(1)$  be the first connective Morava  $K$ -theory. It has a unique  $S$ -algebra structure [Ang, Th. A], and we show in Section 2 that  $\ell/p$  is an  $\ell_p$ -algebra (in uncountably many ways), so that  $K(\ell/p)$  is a  $K(\ell_p)$ -module spectrum.

Let  $V(1) = S/(p, v_1)$  be the type 2 Smith–Toda complex (see Section 4 below for a definition). It is a homotopy commutative ring spectrum for  $p \geq 5$ , with a preferred periodic class  $v_2 \in V(1)_*$  of degree  $2p^2 - 2$ . We write  $V(1)_*(X) = \pi_*(V(1) \wedge X)$  for the  $V(1)$ -homotopy of a spectrum  $X$ . Multiplication by  $v_2$  makes  $V(1)_*(X)$  a  $P(v_2)$ -module, where  $P(v_2)$  denotes the polynomial algebra over  $\mathbb{F}_p$  generated by  $v_2$ . We denote by  $\mathbb{F}_p\{x_1, \dots, x_n\}$  the  $\mathbb{F}_p$ -vector space generated by  $x_1, \dots, x_n$ , and by  $E(x_1, \dots, x_n)$  the exterior algebra over  $\mathbb{F}_p$  generated by  $x_1, \dots, x_n$ .

We computed the  $V(1)$ -homotopy of  $K(\ell_p)$  in [AR02, Th. 9.1], showing that it is essentially a free  $P(v_2)$ -module on  $(4p + 4)$  generators. The following is our main result, corresponding to Theorem 7.7 in the body of the paper.

**Theorem 1.1.** *Let  $p \geq 5$  be a prime and let  $\ell/p = k(1)$  be the first connective Morava  $K$ -theory spectrum. There is an isomorphism of  $P(v_2)$ -modules*

$$\begin{aligned} V(1)_*K(\ell/p) &\cong P(v_2) \otimes E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{1, \partial\lambda_2, \lambda_2, \partial v_2\} \\ &\quad \oplus P(v_2) \otimes E(\mathrm{dlog} v_1) \otimes \mathbb{F}_p\{t^d v_2 \mid 0 < d < p^2 - p, p \nmid d\} \\ &\quad \oplus P(v_2) \otimes E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^{dp} \lambda_2 \mid 0 < d < p\}. \end{aligned}$$

Here  $|\lambda_1| = |\bar{\epsilon}_1| = 2p - 1$ ,  $|\lambda_2| = 2p^2 - 1$ ,  $|v_2| = 2p^2 - 2$ ,  $|\mathrm{dlog} v_1| = 1$ ,  $|\partial| = -1$  and  $|t| = -2$ . This is a free  $P(v_2)$ -module of rank  $(2p^2 - 2p + 8)$  and of zero Euler characteristic.

We prove this theorem by means of the cyclotomic trace map [BHM93] to topological cyclic homology  $TC(\ell/p; p)$ . Along the way we evaluate  $V(1)_*THH(\ell/p)$ , where  $THH$  denotes topological Hochschild homology, as well as  $V(1)_*TC(\ell/p; p)$ , see Proposition 4.2 and Theorem 7.6.

Let  $L_p$  be the  $p$ -complete Adams summand of periodic complex  $K$ -theory, and let  $L/p = K(1)$  be the first periodic Morava  $K$ -theory. The localization cofiber sequence

$K(\mathbb{Z}_p) \rightarrow K(\ell_p) \rightarrow K(L_p) \rightarrow \Sigma K(\mathbb{Z}_p)$  of Blumberg and Mandell [BM08, p. 157] has the mod  $p$  Adams analogue

$$K(\mathbb{Z}/p) \rightarrow K(\ell/p) \rightarrow K(L/p) \rightarrow \Sigma K(\mathbb{Z}/p),$$

see Proposition 2.2 below. Using Quillen's computation [Qui72, Th. 7] of  $K(\mathbb{Z}/p)$ , we obtain the following consequence:

**Corollary 1.2.** *Let  $p \geq 5$  be a prime and let  $L/p = K(1)$  be the first Morava  $K$ -theory spectrum. There is an isomorphism of  $P(v_2^{\pm 1})$ -modules*

$$V(1)_*K(L/p)[v_2^{-1}] \cong V(1)_*K(\ell/p)[v_2^{-1}].$$

*If there is a class  $\mathrm{dlog} v_1 \in V(1)_1K(L/p)$  with  $\lambda_2 = v_2 \cdot \mathrm{dlog} v_1$ , then there is an isomorphism of  $P(v_2)$ -modules*

$$\begin{aligned} V(1)_*K(L/p) &\cong P(v_2) \otimes E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{1, \partial\lambda_2, \mathrm{dlog} v_1, \partial v_2\} \\ &\quad \oplus P(v_2) \otimes E(\mathrm{dlog} v_1) \otimes \mathbb{F}_p\{t^d v_2 \mid 0 < d < p^2 - p, p \nmid d\} \\ &\quad \oplus P(v_2) \otimes E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^{dp} v_2 \mathrm{dlog} v_1 \mid 0 < d < p\}, \end{aligned}$$

*where the degrees of the generators are as in Theorem 1.1. This is a free  $P(v_2)$ -module of rank  $(2p^2 - 2p + 8)$  and of zero Euler characteristic.*

Our far-reaching aim, which partially motivated the computations presented here, is to conceptually understand the algebraic  $K$ -theory of  $\ell_p$  and other commutative  $S$ -algebras in terms of localization and Galois descent, in the same way as we understand the algebraic  $K$ -theory of rings of integers in (local) number fields or more general regular rings. The first task is to relate  $K(\ell_p)$  to the algebraic  $K$ -theory of its “residue fields” and “fraction field”, for which we expect a description in terms of Galois cohomology to exist, starting with the Galois theory for commutative  $S$ -algebras developed by the second author [Rog08]. The residue rings of  $\ell_p$  appear to be  $\ell/p$ ,  $H\mathbb{Z}_p$  and  $H\mathbb{Z}/p$ , while the fraction field  $\mathcal{ff}(\ell_p)$  is more mysterious. For our purposes, its algebraic  $K$ -theory  $K(\mathcal{ff}(\ell_p))$  should fit in a natural localization cofibre sequence of spectra

$$K(L/p) \rightarrow K(L_p) \rightarrow K(\mathcal{ff}(\ell_p)) \rightarrow \Sigma K(L/p).$$

An obvious candidate for  $\mathcal{ff}(\ell_p)$  is provided by the algebraic localization  $L_p[p^{-1}] = L\mathbb{Q}_p$ , having as coefficients the graded field  $\mathbb{Q}_p[v_1^{\pm 1}]$ . However, by the following corollary, this is too naive.

**Corollary 1.3.** *The spectra  $K(L/p)$ ,  $K(L_p)$  and  $K(L\mathbb{Q}_p)$  cannot possibly fit in a cofibre sequence*

$$K(L/p) \rightarrow K(L_p) \rightarrow K(L\mathbb{Q}_p) \rightarrow \Sigma K(L/p).$$

Indeed, the above computation implies that  $V(1)_*K(L/p)[v_2^{-1}]$  and  $V(1)_*K(L_p)[v_2^{-1}]$  are not abstractly isomorphic, while  $V(1)_*K(L\mathbb{Q}_p)[v_2^{-1}]$  is zero since it is an algebra over  $V(1)_*K(\mathbb{Q}_p)[v_2^{-1}] = 0$ . The later equality follows from the computation of the  $p$ -primary homotopy type of  $K(\mathbb{Q}_p)$  [HM03, Th. D], which shows that  $V(1)_*K(\mathbb{Q}_p)$  is  $v_2$ -torsion.

In conclusion, the conjectural fraction field  $\mathcal{ff}(\ell_p)$  appears to be a localization of  $L_p$  away from  $L/p$  less drastic than the algebraic localization  $L_p[p^{-1}] = L\mathbb{Q}_p$ . We elaborate more on this issue in [AR].

The paper is organized as follows. In Section 2 we fix our notations, show that  $\ell/p$  admits the structure of an associative  $\ell_p$ -algebra, and give a similar discussion for  $ku/p$

and the periodic versions  $L/p$  and  $KU/p$ . Section 3 contains the computation of the mod  $p$  homology of  $THH(\ell/p)$ , and in Section 4 we evaluate its  $V(1)$ -homotopy. In Section 5 we show that the  $C_{p^n}$ -fixed points and  $C_{p^n}$ -homotopy fixed points of  $THH(\ell/p)$  are closely related, and use this to inductively determine their  $V(1)$ -homotopy in Section 6. Finally, in Section 7 we achieve the computation of  $TC(\ell/p; p)$  and  $K(\ell/p)$  in  $V(1)$ -homotopy.

*Notations and conventions.* Let  $p$  be a fixed prime. We write  $E(x) = \mathbb{F}_p[x]/(x^2)$  for the exterior algebra,  $P(x) = \mathbb{F}_p[x]$  for the polynomial algebra and  $P(x^{\pm 1}) = \mathbb{F}_p[x, x^{-1}]$  for the Laurent polynomial algebra on one generator  $x$ , and similarly for a list of generators. We will also write  $\Gamma(x) = \mathbb{F}_p\{\gamma_i(x) \mid i \geq 0\}$  for the divided power algebra, with  $\gamma_i(x) \cdot \gamma_j(x) = \binom{i+j}{i} \gamma_{i+j}(x)$ , where  $\binom{i+j}{i} = (i+j)!/i!j!$  is the binomial coefficient. We use the obvious abbreviations  $\gamma_0(x) = 1$  and  $\gamma_1(x) = x$ . Finally, we write  $P_h(x) = \mathbb{F}_p[x]/(x^h)$  for the truncated polynomial algebra of height  $h$ , and recall the isomorphism  $\Gamma(x) \cong P_p(\gamma_{p^e}(x) \mid e \geq 0)$  in characteristic  $p$ . We write  $X_{(p)}$  and  $X_p$  for the  $p$ -localization and the  $p$ -completion, respectively, of any spectrum or abelian group  $X$ . In the spectral sequences (of  $\mathbb{F}_p$ -modules) discussed below, we often determine differentials only up to multiplication by a unit. We use the notation  $d(x) \doteq y$  to indicate that the equation  $d(x) = \alpha y$  holds for some unit  $\alpha \in \mathbb{F}_p$ .

## 2. BASE CHANGE SQUARES OF $S$ -ALGEBRAS

Let  $p$  be a prime, even or odd for now. Let  $ku$  and  $KU$  be the connective and the periodic complex  $K$ -theory spectra, with homotopy rings  $ku_* = \mathbb{Z}[u]$  and  $KU_* = \mathbb{Z}[u^{\pm 1}]$ , where  $|u| = 2$ . Let  $\ell = BP\langle 1 \rangle$  and  $L = E(1)$  be the  $p$ -local Adams summands, with  $\ell_* = \mathbb{Z}_{(p)}[v_1]$  and  $L_* = \mathbb{Z}_{(p)}[v_1^{\pm 1}]$ , where  $|v_1| = 2p - 2$ . The inclusion  $\ell \rightarrow ku_{(p)}$  maps  $v_1$  to  $u^{p-1}$ . Alternate notations in the  $p$ -complete cases are  $KU_p = E_1$  and  $L_p = \widehat{E(1)}$ . These ring spectra are all commutative  $S$ -algebras, in the sense that each admits a unique  $E_\infty$  ring spectrum structure. See [BR05, p. 692] for proofs of uniqueness in the periodic cases.

Let  $ku/p$  and  $KU/p$  be the connective and periodic mod  $p$  complex  $K$ -theory spectra, with coefficients  $(ku/p)_* = \mathbb{Z}/p[u]$  and  $(KU/p)_* = \mathbb{Z}/p[u^{\pm 1}]$ . These are 2-periodic versions of the first Morava  $K$ -theory spectra  $\ell/p = k(1)$  and  $L/p = K(1)$ , with  $(\ell/p)_* = \mathbb{Z}/p[v_1]$  and  $(L/p)_* = \mathbb{Z}/p[v_1^{\pm 1}]$ . Each of these can be constructed as the cofiber of the multiplication by  $p$  map, as a module over the corresponding commutative  $S$ -algebra. For example, there is a cofiber sequence of  $ku$ -modules  $ku \xrightarrow{p} ku \xrightarrow{i} ku/p \rightarrow \Sigma ku$ .

Let  $HR$  be the Eilenberg–Mac Lane spectrum of a ring  $R$ . When  $R$  is associative,  $HR$  admits a unique associative  $S$ -algebra structure, and when  $R$  is commutative,  $HR$  admits a unique commutative  $S$ -algebra structure. The zeroth Postnikov section defines unique maps of commutative  $S$ -algebras  $\pi: ku \rightarrow H\mathbb{Z}$  and  $\pi: \ell \rightarrow H\mathbb{Z}_{(p)}$ , which can be followed by unique commutative  $S$ -algebra maps to  $H\mathbb{Z}/p$ .

The  $ku$ -module spectrum  $ku/p$  does not admit the structure of a commutative  $ku$ -algebra. It cannot even be an  $E_2$  or  $H_2$  ring spectrum, since the homomorphism induced in mod  $p$  homology by the resulting map  $\pi: ku/p \rightarrow H\mathbb{Z}/p$  of  $H_2$  ring spectra would not commute with the homology operation  $Q^1(\bar{\tau}_0) = \bar{\tau}_1$  in the target  $H_*(H\mathbb{Z}/p; \mathbb{F}_p)$  [BMMS86, III.2.3]. Similar remarks apply for  $KU/p$ ,  $\ell/p$  and  $L/p$ . Associative algebra structures, or  $A_\infty$  ring spectrum structures, are easier to come by. The following result is a direct application of the methods of [Laz01, §§9–11]. We adapt the notation of [BJ02, §3] to provide some details in our case.

**Proposition 2.1.** *The  $ku$ -module spectrum  $ku/p$  admits the structure of an associative  $ku$ -algebra, but the structure is not unique. Similar statements hold for  $KU/p$  as a  $KU$ -algebra,  $\ell/p$  as an  $\ell$ -algebra and  $L/p$  as an  $L$ -algebra.*

*Proof.* We construct  $ku/p$  as the (homotopy) limit of its Postnikov tower of associative  $ku$ -algebras  $P^{2m-2} = ku/(p, u^m)$ , with coefficient rings  $ku/(p, u^m)_* = ku_*/(p, u^m)$  for  $m \geq 1$ . To start the induction,  $P^0 = H\mathbb{Z}/p$  is a  $ku$ -algebra via  $i \circ \pi: ku \rightarrow H\mathbb{Z} \rightarrow H\mathbb{Z}/p$ . Assume inductively for  $m \geq 1$  that  $P = P^{2m-2}$  has been constructed. We will define  $P^{2m}$  by a (homotopy) pullback diagram

$$\begin{array}{ccc} P^{2m} & \xrightarrow{\quad} & P \\ \downarrow & & \downarrow \text{in}_1 \\ P & \xrightarrow{\quad d \quad} & P \vee \Sigma^{2m+1} H\mathbb{Z}/p \end{array}$$

in the category of associative  $ku$ -algebras. Here

$$d \in \text{ADer}_{ku}^{2m+1}(P, H\mathbb{Z}/p) \cong THH_{ku}^{2m+2}(P, H\mathbb{Z}/p)$$

is an associative  $ku$ -algebra derivation of  $P$  with values in  $\Sigma^{2m+1} H\mathbb{Z}/p$ , and the group of such can be identified with the indicated topological Hochschild cohomology group of  $P$  over  $ku$ . We recall that these are the homotopy groups (cohomologically graded) of the function spectrum  $F_{P \wedge_{ku} P^{op}}(P, H\mathbb{Z}/p)$ . The composite map  $pr_2 \circ d: P \rightarrow \Sigma^{2m+1} H\mathbb{Z}/p$  of  $ku$ -modules, where  $pr_2$  projects onto the second wedge summand, is restricted to equal the  $ku$ -module Postnikov  $k$ -invariant of  $ku/p$  in

$$H_{ku}^{2m+1}(P; \mathbb{Z}/p) = \pi_0 F_{ku}(P, \Sigma^{2m+1} H\mathbb{Z}/p).$$

We compute that  $\pi_*(P \wedge_{ku} P^{op}) = ku_*/(p, u^m) \otimes E(\tau_0, \tau_{1,m})$ , where  $|\tau_0| = 1$ ,  $|\tau_{1,m}| = 2m+1$  and  $E(-)$  denotes the exterior algebra on the given generators. (For  $p = 2$ , the use of the opposite product is essential here [Ang08, §3].) The function spectrum description of topological Hochschild cohomology leads to the spectral sequence

$$\begin{aligned} E_2^{*,*} &= \text{Ext}_{\pi_*(P \wedge_{ku} P^{op})}^{*,*}(\pi_*(P), \mathbb{Z}/p) \\ &\cong \mathbb{Z}/p[y_0, y_{1,m}] \\ &\implies THH_{ku}^*(P, H\mathbb{Z}/p), \end{aligned}$$

where  $y_0$  and  $y_{1,m}$  have cohomological bidegrees  $(1, 1)$  and  $(1, 2m+1)$ , respectively. The spectral sequence collapses at  $E_2 = E_\infty$ , since it is concentrated in even total degrees. In particular,

$$\text{ADer}_{ku}^{2m+1}(P, H\mathbb{Z}/p) \cong \mathbb{F}_p\{y_{1,m}, y_0^{m+1}\}.$$

Additively,  $H_{ku}^{2m+1}(P; \mathbb{Z}/p) \cong \mathbb{F}_p\{Q_{1,m}\}$  is generated by a class dual to  $\tau_{1,m}$ , which is the image of  $y_{1,m}$  under left composition with  $pr_2$ . It equals the  $ku$ -module  $k$ -invariant of  $ku/p$ . Thus there are precisely  $p$  choices  $d = y_{1,m} + \alpha y_0^{m+1}$ , with  $\alpha \in \mathbb{F}_p$ , for how to extend any given associative  $ku$ -algebra structure on  $P = P^{2m-2}$  to one on  $P^{2m} = ku/(p, u^{m+1})$ . In the limit, we find that there are an uncountable number of associative  $ku$ -algebra structures on  $ku/p = \text{holim}_m P^{2m}$ , each indexed by a sequence of choices  $\alpha \in \mathbb{F}_p$  for all  $m \geq 1$ .

The periodic spectrum  $KU/p$  can be obtained from  $ku/p$  by Bousfield  $KU$ -localization in the category of  $ku$ -modules [EKMM97, VIII.4], which makes it an associative  $KU$ -algebra. The classification of periodic  $S$ -algebra structures is the same as in the connective case, since the original  $ku$ -algebra structure on  $ku/p$  can be recovered from

that on  $KU/p$  by a functorial passage to the connective cover. To construct  $\ell/p$  as an associative  $\ell$ -algebra, or  $L/p$  as an associative  $L$ -algebra, replace  $u$  by  $v_1$  in these arguments.  $\square$

By varying the ground  $S$ -algebra, we obtain the same conclusions about  $ku/p$  as a  $ku_{(p)}$ -algebra or  $ku_p$ -algebra, and about  $\ell/p$  as an  $\ell_p$ -algebra.

For each choice of  $ku$ -algebra structure on  $ku/p$ , the zeroth Postnikov section

$$\pi: ku/p \rightarrow H\mathbb{Z}/p$$

is a  $ku$ -algebra map, with the unique  $ku$ -algebra structure on the target. Hence there is a commutative square of associative  $ku$ -algebras

$$\begin{array}{ccc} ku & \xrightarrow{i} & ku/p \\ \downarrow \pi & & \downarrow \pi \\ H\mathbb{Z} & \xrightarrow{i} & H\mathbb{Z}/p \end{array}$$

and similarly in the  $p$ -local and  $p$ -complete cases. In view of the weak equivalence  $H\mathbb{Z} \wedge_{ku} ku/p \simeq H\mathbb{Z}/p$ , this square expresses the associative  $H\mathbb{Z}$ -algebra  $H\mathbb{Z}/p$  as the base change of the associative  $ku$ -algebra  $ku/p$  along  $\pi: ku \rightarrow H\mathbb{Z}$ . Likewise, there is a commutative square of associative  $\ell_p$ -algebras

$$\begin{array}{ccc} \ell_p & \xrightarrow{i} & \ell/p \\ \downarrow \pi & & \downarrow \pi \\ H\mathbb{Z}_p & \xrightarrow{i} & H\mathbb{Z}/p \end{array} \tag{2.1}$$

that expresses  $H\mathbb{Z}/p$  as the base change of  $\ell/p$  along  $\ell_p \rightarrow H\mathbb{Z}_p$ , and similarly in the  $p$ -local case. By omission of structure, these squares are also diagrams of  $S$ -algebras and  $S$ -algebra maps.

We end this section by formulating the mod  $p$  analogue of the localization cofibre sequence in algebraic  $K$ -theory

$$K(\mathbb{Z}_p) \rightarrow K(\ell_p) \rightarrow K(L_p) \rightarrow \Sigma K(\mathbb{Z}_p) \tag{2.2}$$

conjectured by the second author and established by Blumberg and Mandell [BM08, p. 157]

**Proposition 2.2.** *There is a localization cofibre sequence of spectra*

$$K(\mathbb{Z}/p) \rightarrow K(\ell/p) \rightarrow K(L/p) \rightarrow \Sigma K(\mathbb{Z}/p)$$

where the first map is the transfer and the second map is induced by the localization  $\ell/p \rightarrow \ell/p[v_1^{-1}] = L/p$ .

*Proof.* The proof of the existence of the localization sequence (2.2) given in [BM08, p. 160–163] and the identification of the transfer map adapt without change to cover the mod  $p$  analogue stated in this proposition. Here we use that a finite cell  $\ell/p$ -module that is  $v_1$ -torsion has finite homotopy groups, and the non-zero groups are concentrated in a finite range of degrees.  $\square$

## 3. TOPOLOGICAL HOCHSCHILD HOMOLOGY

We shall compute the  $V(1)$ -homotopy of the topological Hochschild homology  $THH(-)$  and topological cyclic homology  $TC(-; p)$  of the  $S$ -algebras in diagram (2.1), for primes  $p \geq 5$ . Passing to connective covers, this also computes the  $V(1)$ -homotopy of the algebraic  $K$ -theory spectra appearing in that square. With these coefficients, or more generally, after  $p$ -adic completion, the functors  $THH$  and  $TC$  are insensitive to  $p$ -completion in the argument, so we shall simplify the notation slightly by working with the associative  $S$ -algebras  $\ell$  and  $H\mathbb{Z}_{(p)}$  in place of  $\ell_p$  and  $H\mathbb{Z}_p$ . For ordinary rings  $R$  we almost always shorten notations like  $THH(HR)$  to  $THH(R)$ .

The computations follow the strategy of [Bök], [BM94], [BM95] and [HM97] for  $H\mathbb{Z}/p$  and  $H\mathbb{Z}$ , and of [MS93] and [AR02] for  $\ell$ . See also [AR05, §§4–7] for further discussion of the  $THH$ -part of such computations. In this section we shall compute the mod  $p$  homology of the topological Hochschild homology of  $\ell/p$  as a module over the corresponding homology for  $\ell$ , for any odd prime  $p$ .

*Remark 3.1.* Our computations are based on comparisons, using the maps displayed in diagram (2.1) above. We will abuse notation and use the same name for classes in the homology or  $V(1)$ -homotopy of  $THH(\ell_p)$ ,  $THH(\ell/p)$ ,  $THH(\mathbb{Z}_p)$  or  $THH(\mathbb{Z}/p)$ , when these classes unambiguously correspond to each other under the homomorphisms induced by the maps  $i$  and  $\pi$  in (2.1). We also use this abuse of notations in later sections for the  $V(1)$ -homotopy of  $TC$ , etc.

We write  $H_*(-)$  for homology with mod  $p$  coefficients. It takes values in graded  $A_*$ -comodules, where  $A_*$  is the dual Steenrod algebra [Mil58, Th 2]. Explicitly (for  $p$  odd),

$$A_* = P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k \geq 0)$$

with coproduct

$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{p^i}$$

and

$$\psi(\bar{\tau}_k) = 1 \otimes \bar{\tau}_k + \sum_{i+j=k} \bar{\tau}_i \otimes \bar{\xi}_j^{p^i}.$$

Here  $\bar{\xi}_0 = 1$ ,  $\bar{\xi}_k = \chi(\xi_k)$  has degree  $2(p^k - 1)$  and  $\bar{\tau}_k = \chi(\tau_k)$  has degree  $2p^k - 1$ , where  $\chi$  is the canonical conjugation [MM65, 8.4]. Then the maps  $i$  and the zeroth Postnikov sections  $\pi$  of (2.1) induce identifications

$$\begin{aligned} H_*(H\mathbb{Z}_{(p)}) &= P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k \geq 1) \\ H_*(\ell) &= P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k \geq 2) \\ H_*(\ell/p) &= P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_0, \bar{\tau}_k \mid k \geq 2) \end{aligned}$$

as  $A_*$ -comodule subalgebras of  $H_*(H\mathbb{Z}/p) = A_*$ . We often make use of the following  $A_*$ -comodule coactions

$$\begin{aligned} \nu(\bar{\tau}_0) &= 1 \otimes \bar{\tau}_0 + \bar{\tau}_0 \otimes 1 \\ \nu(\bar{\xi}_1) &= 1 \otimes \bar{\xi}_1 + \bar{\xi}_1 \otimes 1 \\ \nu(\bar{\tau}_1) &= 1 \otimes \bar{\tau}_1 + \bar{\tau}_0 \otimes \bar{\xi}_1 + \bar{\tau}_1 \otimes 1 \\ \nu(\bar{\xi}_2) &= 1 \otimes \bar{\xi}_2 + \bar{\xi}_1 \otimes \bar{\xi}_1^p + \bar{\xi}_2 \otimes 1 \\ \nu(\bar{\tau}_2) &= 1 \otimes \bar{\tau}_2 + \bar{\tau}_0 \otimes \bar{\xi}_2 + \bar{\tau}_1 \otimes \bar{\xi}_1^p + \bar{\tau}_2 \otimes 1. \end{aligned}$$

The Bökstedt spectral sequences

$$E^2(B) = HH_*(H_*(B)) \implies H_*(THH(B))$$

for the commutative  $S$ -algebras  $B = H\mathbb{Z}/p$ ,  $H\mathbb{Z}_{(p)}$  and  $\ell$  begin

$$\begin{aligned} E^2(\mathbb{Z}/p) &= A_* \otimes E(\sigma\bar{\xi}_k \mid k \geq 1) \otimes \Gamma(\sigma\bar{\tau}_k \mid k \geq 0) \\ E^2(\mathbb{Z}_{(p)}) &= H_*(H\mathbb{Z}_{(p)}) \otimes E(\sigma\bar{\xi}_k \mid k \geq 1) \otimes \Gamma(\sigma\bar{\tau}_k \mid k \geq 1) \\ E^2(\ell) &= H_*(\ell) \otimes E(\sigma\bar{\xi}_k \mid k \geq 1) \otimes \Gamma(\sigma\bar{\tau}_k \mid k \geq 2). \end{aligned}$$

Here  $HH_*(H_*(B))$  denotes the Hochschild homology of the graded  $\mathbb{F}_p$ -algebra  $H_*(B)$ . In the above formula we made use of the  $\mathbb{F}_p$ -linear operator  $\sigma: H_*(B) \rightarrow HH_1(H_*(B))$ ,  $x \mapsto \sigma x$ , where  $\sigma x$  is the class represented by  $1 \otimes x - x \otimes 1$  in the Hochschild complex. Notice that  $\sigma$  is the restriction of Connes' operator  $d$  to  $HH_0(H_*(B)) = H_*(B)$ , and is a derivation in the sense that

$$\sigma(xy) = x\sigma(y) + (-1)^{|x||y|}y\sigma(x)$$

for all  $x, y \in H_*(B)$ . These spectral sequences are (graded) commutative  $A_*$ -comodule algebra spectral sequences, and there are differentials

$$d^{p-1}(\gamma_j \sigma\bar{\tau}_k) \doteq \sigma\bar{\xi}_{k+1} \cdot \gamma_{j-p} \sigma\bar{\tau}_k$$

for  $j \geq p$  and  $k \geq 0$ , see [Bök, Lem. 1.3], [Hun96, Th. 1] or [Aus05, Lem. 5.3], leaving

$$\begin{aligned} E^\infty(\mathbb{Z}/p) &= A_* \otimes P_p(\sigma\bar{\tau}_k \mid k \geq 0) \\ E^\infty(\mathbb{Z}_{(p)}) &= H_*(H\mathbb{Z}_{(p)}) \otimes E(\sigma\bar{\xi}_1) \otimes P_p(\sigma\bar{\tau}_k \mid k \geq 1) \\ E^\infty(\ell) &= H_*(\ell) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2) \otimes P_p(\sigma\bar{\tau}_k \mid k \geq 2). \end{aligned}$$

The inclusion of 0-simplices  $\eta: B \rightarrow THH(B)$  is split for commutative  $B$  by the augmentation  $\epsilon: THH(B) \rightarrow B$ . Thus there are unique representatives in Bökstedt filtration 1, with zero augmentation, for each of the classes  $\sigma x$ . There are multiplicative extensions  $(\sigma\bar{\tau}_k)^p = \sigma\bar{\tau}_{k+1}$  for  $k \geq 0$ , see [AR05, Prop. 5.9], so

$$\begin{aligned} H_*(THH(\mathbb{Z}/p)) &= A_* \otimes P(\sigma\bar{\tau}_0) \\ H_*(THH(\mathbb{Z}_{(p)})) &= H_*(H\mathbb{Z}_{(p)}) \otimes E(\sigma\bar{\xi}_1) \otimes P(\sigma\bar{\tau}_1) \\ H_*(THH(\ell)) &= H_*(\ell) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2) \otimes P(\sigma\bar{\tau}_2) \end{aligned}$$

as  $A_*$ -comodule algebras. The  $A_*$ -comodule coactions are given by

$$\begin{aligned} \nu(\sigma\bar{\tau}_0) &= 1 \otimes \sigma\bar{\tau}_0 \\ \nu(\sigma\bar{\xi}_1) &= 1 \otimes \sigma\bar{\xi}_1 \\ \nu(\sigma\bar{\tau}_1) &= 1 \otimes \sigma\bar{\tau}_1 + \bar{\tau}_0 \otimes \sigma\bar{\xi}_1 \\ \nu(\sigma\bar{\xi}_2) &= 1 \otimes \sigma\bar{\xi}_2 \\ \nu(\sigma\bar{\tau}_2) &= 1 \otimes \sigma\bar{\tau}_2 + \bar{\tau}_0 \otimes \sigma\bar{\xi}_2. \end{aligned} \tag{3.1}$$

The natural map  $\pi_*: THH(\ell) \rightarrow THH(\mathbb{Z}_{(p)})$  induced by  $\pi: \ell \rightarrow \mathbb{Z}_{(p)}$  takes  $\sigma\bar{\xi}_2$  to 0 and  $\sigma\bar{\tau}_2$  to  $(\sigma\bar{\tau}_1)^p$ . The natural map  $i_*: THH(\mathbb{Z}_{(p)}) \rightarrow THH(\mathbb{Z}/p)$  induced by  $i: \mathbb{Z}_{(p)} \rightarrow \mathbb{Z}/p$  takes  $\sigma\bar{\xi}_1$  to 0 and  $\sigma\bar{\tau}_1$  to  $(\sigma\bar{\tau}_0)^p$ .

The Bökstedt spectral sequence for the associative  $S$ -algebra  $B = \ell/p$  begins

$$E^2(\ell/p) = H_*(\ell/p) \otimes E(\sigma\bar{\xi}_k \mid k \geq 1) \otimes \Gamma(\sigma\bar{\tau}_0, \sigma\bar{\tau}_k \mid k \geq 2).$$

It is an  $A_*$ -comodule module spectral sequence over the Bökstedt spectral sequence for  $\ell$ , since the  $\ell$ -algebra multiplication  $\ell \wedge \ell/p \rightarrow \ell/p$  is a map of associative  $S$ -algebras. However, it is not itself an algebra spectral sequence, since the product on  $\ell/p$  is not commutative enough to induce a natural product structure on  $THH(\ell/p)$ . Nonetheless, we will use the algebra structure present at the  $E^2$ -term to help in naming classes.

The map  $\pi: \ell/p \rightarrow H\mathbb{Z}/p$  induces an injection of Bökstedt spectral sequence  $E^2$ -terms, so there are differentials generated algebraically by

$$d^{p-1}(\gamma_j \sigma \bar{\tau}_k) \doteq \sigma \bar{\xi}_{k+1} \cdot \gamma_{j-p} \sigma \bar{\tau}_k$$

for  $j \geq p$ ,  $k = 0$  or  $k \geq 2$ , leaving

$$E^\infty(\ell/p) = H_*(\ell/p) \otimes E(\sigma \bar{\xi}_2) \otimes P_p(\sigma \bar{\tau}_0, \sigma \bar{\tau}_k \mid k \geq 2) \quad (3.2)$$

as an  $A_*$ -comodule module over  $E^\infty(\ell)$ . In order to obtain  $H_*(THH(\ell/p))$ , we need to resolve the  $A_*$ -comodule and  $H_*(THH(\ell))$ -module extensions. This is achieved in Lemma 3.3 below.

The natural map  $\pi_*: E^\infty(\ell/p) \rightarrow E^\infty(\mathbb{Z}/p)$  is an isomorphism in total degrees  $\leq (2p-2)$  and injective in total degrees  $\leq (2p^2-2)$ . The first class in the kernel is  $\sigma \bar{\xi}_2$ . Hence there are unique classes

$$1, \bar{\tau}_0, \sigma \bar{\tau}_0, \bar{\tau}_0 \sigma \bar{\tau}_0, \dots, (\sigma \bar{\tau}_0)^{p-1}$$

in degrees  $0 \leq * \leq 2p-2$  of  $H_*(THH(\ell/p))$ , mapping to classes with the same names in  $H_*(THH(\mathbb{Z}/p))$ . More concisely, these are the monomials  $\bar{\tau}_0^\delta (\sigma \bar{\tau}_0)^i$  for  $0 \leq \delta \leq 1$  and  $0 \leq i \leq p-1$ , except that the degree  $(2p-1)$  case  $(\delta, i) = (1, p-1)$  is omitted. The  $A_*$ -comodule coaction on these classes is given by the same formulas in  $H_*(THH(\ell/p))$  as in  $H_*(THH(\mathbb{Z}/p))$ , cf. (3.1).

There is also a class  $\bar{\xi}_1$  in degree  $(2p-2)$  of  $H_*(THH(\ell/p))$  mapping to a class with the same name, and same  $A_*$ -coaction, in  $H_*(THH(\mathbb{Z}/p))$ .

In degree  $(2p-1)$ ,  $\pi_*$  is a map of extensions from

$$0 \rightarrow \mathbb{F}_p\{\bar{\xi}_1 \bar{\tau}_0\} \rightarrow H_{2p-1}(THH(\ell/p)) \rightarrow \mathbb{F}_p\{\bar{\tau}_0 (\sigma \bar{\tau}_0)^{p-1}\} \rightarrow 0$$

to

$$0 \rightarrow \mathbb{F}_p\{\bar{\tau}_1, \bar{\xi}_1 \bar{\tau}_0\} \rightarrow H_{2p-1}(THH(\mathbb{Z}/p)) \rightarrow \mathbb{F}_p\{\bar{\tau}_0 (\sigma \bar{\tau}_0)^{p-1}\} \rightarrow 0.$$

The latter extension is canonically split by the augmentation  $\epsilon: THH(\mathbb{Z}/p) \rightarrow H\mathbb{Z}/p$ , which uses the commutativity of the  $S$ -algebra  $H\mathbb{Z}/p$ .

In degree  $2p$ , the map  $\pi_*$  goes from

$$H_{2p}(THH(\ell/p)) = \mathbb{F}_p\{\bar{\xi}_1 \sigma \bar{\tau}_0\}$$

to

$$0 \rightarrow \mathbb{F}_p\{\bar{\tau}_0 \bar{\tau}_1\} \rightarrow H_{2p}(THH(\mathbb{Z}/p)) \rightarrow \mathbb{F}_p\{\sigma \bar{\tau}_1, \bar{\xi}_1 \sigma \bar{\tau}_0\} \rightarrow 0.$$

Again the latter extension is canonically split.

**Lemma 3.2.** *There is a unique class  $y$  in  $H_{2p-1}(THH(\ell/p))$  represented by  $\bar{\tau}_0 (\sigma \bar{\tau}_0)^{p-1}$  in  $E_{p-1,p}^\infty(\ell/p)$  and mapped by  $\pi_*$  to  $\bar{\tau}_0 (\sigma \bar{\tau}_0)^{p-1} - \bar{\tau}_1$  in  $H_*(THH(\mathbb{Z}/p))$ .*

*Proof.* This follows from naturality of the suspension operator  $\sigma$  and the multiplicative relation  $(\sigma \bar{\tau}_0)^p = \sigma \bar{\tau}_1$  in  $H_*(THH(\mathbb{Z}/p))$ . A class  $y$  in  $H_{2p-1}(THH(\ell/p))$  represented by  $\bar{\tau}_0 (\sigma \bar{\tau}_0)^{p-1}$  is determined modulo  $\bar{\xi}_1 \bar{\tau}_0$ . Its image in  $H_{2p-1}(THH(\mathbb{Z}/p))$  thus has the form  $\alpha \bar{\tau}_1 + \bar{\tau}_0 (\sigma \bar{\tau}_0)^{p-1}$  modulo  $\bar{\xi}_1 \bar{\tau}_0$ , for some  $\alpha \in \mathbb{F}_p$ . The suspension  $\sigma y$  lies in  $H_{2p}(THH(\ell/p)) = \mathbb{F}_p\{\bar{\xi}_1 \sigma \bar{\tau}_0\}$ , so its image in  $H_{2p}(THH(\mathbb{Z}/p))$  is 0 modulo  $\bar{\tau}_0 \bar{\tau}_1$  and  $\bar{\xi}_1 \sigma \bar{\tau}_0$ . It is also the suspension of  $\alpha \bar{\tau}_1 + \bar{\tau}_0 (\sigma \bar{\tau}_0)^{p-1}$  modulo  $\bar{\xi}_1 \bar{\tau}_0$ , which equals



$\sigma(\alpha\bar{\tau}_1) + (\sigma\bar{\tau}_0)^p = (\alpha + 1)\sigma\bar{\tau}_1$ . In particular, the coefficient  $(\alpha + 1)$  of  $\sigma\bar{\tau}_1$  is 0, so  $\alpha = -1$ .  $\square$

Let

$$H_*(THH(\ell))/(\sigma\bar{\xi}_1) = H_*(\ell) \otimes E(\sigma\bar{\xi}_2) \otimes P(\sigma\bar{\tau}_2)$$

denote the quotient algebra of  $H_*(THH(\ell))$  by the ideal generated by  $\sigma\bar{\xi}_1$ .

**Lemma 3.3.** *The classes*

$$1, \bar{\tau}_0, \sigma\bar{\tau}_0, \bar{\tau}_0\sigma\bar{\tau}_0, \dots, (\sigma\bar{\tau}_0)^{p-1}, \bar{\tau}_0(\sigma\bar{\tau}_0)^{p-1},$$

*in  $E^\infty(\ell/p)$  represent unique homology classes in  $H_*(THH(\ell/p))$ , which by abuse of notation will be denoted*

$$1, \bar{\tau}_0, \sigma\bar{\tau}_0, \bar{\tau}_0\sigma\bar{\tau}_0, \dots, (\sigma\bar{\tau}_0)^{p-1}, y,$$

*mapping under  $\pi_*$  to classes with the same names in  $H_*(THH(\mathbb{Z}/p))$ , except for  $y$ , which maps to*

$$\bar{\tau}_0(\sigma\bar{\tau}_0)^{p-1} - \bar{\tau}_1.$$

*The graded  $H_*(THH(\ell))$ -module  $H_*(THH(\ell/p))$  is a free  $H_*(THH(\ell))/(\sigma\bar{\xi}_1)$ -module of rank  $2p$  generated by these classes in degrees 0 through  $2p - 1$ :*

$$H_*(THH(\ell/p)) = H_*(THH(\ell))/(\sigma\bar{\xi}_1) \otimes \mathbb{F}_p\{1, \bar{\tau}_0, \sigma\bar{\tau}_0, \bar{\tau}_0\sigma\bar{\tau}_0, \dots, (\sigma\bar{\tau}_0)^{p-1}, y\}.$$

*The  $A_*$ -comodule coactions are given by*

$$\nu((\sigma\bar{\tau}_0)^i) = 1 \otimes (\sigma\bar{\tau}_0)^i$$

*for  $0 \leq i \leq p - 1$ ,*

$$\nu(\bar{\tau}_0(\sigma\bar{\tau}_0)^i) = 1 \otimes \bar{\tau}_0(\sigma\bar{\tau}_0)^i + \bar{\tau}_0 \otimes (\sigma\bar{\tau}_0)^i$$

*for  $0 \leq i \leq p - 2$ , and*

$$\nu(y) = 1 \otimes y + \bar{\tau}_0 \otimes (\sigma\bar{\tau}_0)^{p-1} - \bar{\tau}_0 \otimes \bar{\xi}_1 - \bar{\tau}_1 \otimes 1.$$

*Proof.*  $H_*(\ell/p)$  is freely generated as a module over  $H_*(\ell)$  by 1 and  $\bar{\tau}_0$ , and the classes  $\sigma\bar{\xi}_2$  and  $\sigma\bar{\tau}_2$  in  $H_*(THH(\ell))$  induce multiplication by the same symbols in  $E^\infty(\ell/p)$ , as given in (3.2). This generates all of  $E^\infty(\ell/p)$  from the  $2p$  classes  $\bar{\tau}_0^\delta(\sigma\bar{\tau}_0)^i$  for  $0 \leq \delta \leq 1$  and  $0 \leq i \leq p - 1$ .

We claim that multiplication by  $\sigma\bar{\xi}_1$  acts trivially on  $H_*(THH(\ell/p))$ . It suffices to verify this on the module generators  $\bar{\tau}_0^\delta(\sigma\bar{\tau}_0)^i$ , for which the product with  $\sigma\bar{\xi}_1$  remains in the range of degrees where the map to  $H_*(THH(\mathbb{Z}/p))$  is injective. The action of  $\sigma\bar{\xi}_1$  is trivial on  $H_*(THH(\mathbb{Z}/p))$ , since  $d^{p-1}(\gamma_p\sigma\bar{\tau}_0) \doteq \sigma\bar{\xi}_1$  and  $\epsilon(\sigma\bar{\xi}_1) = 0$ , and this implies the claim.

The  $A_*$ -comodule coaction on each module generator, including  $y$ , is determined by that on its image under  $\pi_*$ . In the latter case, for example, we have

$$\begin{aligned} (1 \otimes \pi_*)(\nu(y)) &= \nu(\pi_*(y)) = \nu(\bar{\tau}_0(\sigma\bar{\tau}_0)^{p-1} - \bar{\tau}_1) \\ &= 1 \otimes \bar{\tau}_0(\sigma\bar{\tau}_0)^{p-1} + \bar{\tau}_0 \otimes (\sigma\bar{\tau}_0)^{p-1} - 1 \otimes \bar{\tau}_1 - \bar{\tau}_0 \otimes \bar{\xi}_1 - \bar{\tau}_1 \otimes 1 \\ &= (1 \otimes \pi_*)(1 \otimes y + \bar{\tau}_0 \otimes (\sigma\bar{\tau}_0)^{p-1} - \bar{\tau}_0 \otimes \bar{\xi}_1 - \bar{\tau}_1 \otimes 1), \end{aligned}$$

and this proves our formula for  $\nu(y)$  since  $1 \otimes \pi_*$  is injective in this degree.  $\square$

*Remark 3.4.* Notice that Lemma 3.3 implies that for different choices of  $\ell$ -module structure on  $\ell/p$ , the resulting homology groups  $H_*(THH(\ell/p))$  are (abstractly) isomorphic as graded  $H_*(THH(\ell))$ -modules and  $A_*$ -comodules.

4. PASSAGE TO  $V(1)$ -HOMOTOPY

For  $p \geq 5$  the Smith–Toda complex  $V(1) = S \cup_p e^1 \cup_{\alpha_1} e^{2p-1} \cup_p e^{2p}$  is a homotopy commutative ring spectrum [Smi70, Th 5.1], [Oka84, Ex. 4.5]. It is defined as the mapping cone of the Adams self-map  $v_1: \Sigma^{2p-2}V(0) \rightarrow V(0)$  of the mod  $p$  Moore spectrum  $V(0) = S \cup_p e^1$ . Hence there is a cofiber sequence

$$\Sigma^{2p-2}V(0) \xrightarrow{v_1} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{2p-1}V(0).$$

There are some choices of orientations involved in fixing such an exact triangle, compare for instance with [HM03, Sect. 2.1]. The composite map  $\beta_{1,1} = i_1 j_1: V(1) \rightarrow \Sigma^{2p-1}V(1)$  defines the primary  $v_1$ -Bockstein homomorphism, acting naturally on  $V(1)_*(X)$ .

In this section we compute  $V(1)_*THH(\ell/p)$  as a module over  $V(1)_*THH(\ell)$ , for any prime  $p \geq 5$ . The unique ring spectrum map from  $V(1)$  to  $H\mathbb{Z}/p$  induces the identification

$$H_*(V(1)) = E(\tau_0, \tau_1)$$

(no conjugations) as  $A_*$ -comodule subalgebras of  $A_*$ , see [Tod71, §4]. Here

$$\begin{aligned} \nu(\tau_0) &= 1 \otimes \tau_0 + \tau_0 \otimes 1 \\ \nu(\tau_1) &= 1 \otimes \tau_1 + \xi_1 \otimes \tau_0 + \tau_1 \otimes 1. \end{aligned}$$

A form of the following lemma goes back to [Whi62, p. 271].

**Lemma 4.1.** *Let  $M$  be any  $H\mathbb{Z}/p$ -module spectrum. Then  $M$  is equivalent to a wedge sum of suspensions of  $H\mathbb{Z}/p$ . Hence  $H_*(M)$  is a sum of shifted copies of  $A_*$  as an  $A_*$ -comodule, and the Hurewicz homomorphism  $\pi_*(M) \rightarrow H_*(M)$  identifies  $\pi_*(M)$  with the  $A_*$ -comodule primitives in  $H_*(M)$ .*

*Proof.* The module action map  $\lambda: H\mathbb{Z}/p \wedge M \rightarrow M$  is a retraction, so  $\pi_*(M)$  is a direct summand of  $\pi_*(H\mathbb{Z}/p \wedge M) = H_*(M)$ , hence is a graded  $\mathbb{Z}/p$ -vector space. Choose maps  $\alpha: S^n \rightarrow M$  that represent a basis for this vector space. The wedge sum of the maps

$$\lambda \circ (1 \wedge \alpha): \Sigma^n H\mathbb{Z}/p = H\mathbb{Z}/p \wedge S^n \rightarrow M$$

is the desired  $\pi_*$ -isomorphism  $\bigvee_{\alpha} \Sigma^n H\mathbb{Z}/p \rightarrow M$ .  $\square$

For each  $\ell$ -algebra  $B$ ,  $V(1) \wedge THH(B)$  is a module spectrum over  $V(1) \wedge THH(\ell)$  and thus over  $V(1) \wedge \ell \simeq H\mathbb{Z}/p$ , so  $H_*(V(1) \wedge THH(B))$  is a sum of copies of  $A_*$  as an  $A_*$ -comodule, by Lemma 4.1. In particular,  $V(1)_*THH(B) = \pi_*(V(1) \wedge THH(B))$  is naturally identified with the subgroup of  $A_*$ -comodule primitives in

$$H_*(V(1) \wedge THH(B)) \cong H_*(V(1)) \otimes H_*(THH(B))$$

with the diagonal  $A_*$ -comodule coaction. We write  $v \wedge x$  for the image of  $v \otimes x$  under this identification, with  $v \in H_*(V(1))$  and  $x \in H_*(THH(B))$ . Let

$$\begin{aligned} \epsilon_0 &= 1 \wedge \bar{\tau}_0 + \tau_0 \wedge 1 \\ \epsilon_1 &= 1 \wedge \bar{\tau}_1 + \tau_0 \wedge \bar{\xi}_1 + \tau_1 \wedge 1 \\ \lambda_1 &= 1 \wedge \sigma \bar{\xi}_1 \\ \lambda_2 &= 1 \wedge \sigma \bar{\xi}_2 \\ \mu_0 &= 1 \wedge \sigma \bar{\tau}_0 \\ \mu_1 &= 1 \wedge \sigma \bar{\tau}_1 + \tau_0 \wedge \sigma \bar{\xi}_1 \\ \mu_2 &= 1 \wedge \sigma \bar{\tau}_2 + \tau_0 \wedge \sigma \bar{\xi}_2. \end{aligned} \tag{4.1}$$

These are all  $A_*$ -comodule primitive, when defined, in  $H_*(V(1) \wedge THH(B))$  for  $B = \ell$ ,  $\ell/p$ ,  $H\mathbb{Z}_p$  or  $H\mathbb{Z}/p$  (see Remark 3.1). By a dimension count,

$$\begin{aligned} V(1)_*THH(\mathbb{Z}/p) &= E(\epsilon_0, \epsilon_1) \otimes P(\mu_0) \\ V(1)_*THH(\mathbb{Z}_{(p)}) &= E(\epsilon_1) \otimes E(\lambda_1) \otimes P(\mu_1) \\ V(1)_*THH(\ell) &= E(\lambda_1, \lambda_2) \otimes P(\mu_2) \end{aligned}$$

as commutative  $\mathbb{F}_p$ -algebras. The map  $\pi: \ell \rightarrow H\mathbb{Z}_{(p)}$  takes  $\lambda_2$  to 0 and  $\mu_2$  to  $\mu_1^p$ . The map  $i: H\mathbb{Z}_{(p)} \rightarrow H\mathbb{Z}/p$  takes  $\lambda_1$  to 0 and  $\mu_1$  to  $\mu_0^p$ . Note that  $\mu_2 \in V(1)_{2p^2}THH(\ell)$  was simply denoted  $\mu$  in [AR02].

In degrees  $\leq (2p-2)$  of  $H_*(V(1) \wedge THH(\ell/p))$  the classes

$$\mu_0^i := 1 \wedge (\sigma \bar{\tau}_0)^i \quad (4.2)$$

for  $0 \leq i \leq p-1$  and

$$\epsilon_0 \mu_0^i := 1 \wedge \bar{\tau}_0 (\sigma \bar{\tau}_0)^i + \tau_0 \wedge (\sigma \bar{\tau}_0)^i \quad (4.3)$$

for  $0 \leq i \leq p-2$  are  $A_*$ -comodule primitive, hence lift uniquely to  $V(1)_*THH(\ell/p)$ . These map to the classes  $\epsilon_0^\delta \mu_0^i$  in  $V(1)_*THH(\mathbb{Z}/p)$  for  $0 \leq \delta \leq 1$  and  $0 \leq i \leq p-1$ , except that the degree bound excludes the top case of  $\epsilon_0 \mu_0^{p-1}$ .

In degree  $(2p-1)$  of  $H_*(V(1) \wedge THH(\ell/p))$  we have generators  $1 \wedge \bar{\xi}_1 \bar{\tau}_0$ ,  $\tau_0 \wedge (\sigma \bar{\tau}_0)^{p-1}$ ,  $\tau_0 \wedge \bar{\xi}_1$ ,  $\tau_1 \wedge 1$  and  $1 \wedge y$ . These have coactions

$$\begin{aligned} \nu(1 \wedge \bar{\xi}_1 \bar{\tau}_0) &= 1 \otimes 1 \wedge \bar{\xi}_1 \bar{\tau}_0 + \bar{\tau}_0 \otimes 1 \wedge \bar{\xi}_1 + \bar{\xi}_1 \otimes 1 \wedge \bar{\tau}_0 + \bar{\xi}_1 \bar{\tau}_0 \otimes 1 \wedge 1 \\ \nu(\tau_0 \wedge (\sigma \bar{\tau}_0)^{p-1}) &= 1 \otimes \tau_0 \wedge (\sigma \bar{\tau}_0)^{p-1} + \tau_0 \otimes 1 \wedge (\sigma \bar{\tau}_0)^{p-1} \\ \nu(\tau_0 \wedge \bar{\xi}_1) &= 1 \otimes \tau_0 \wedge \bar{\xi}_1 + \tau_0 \otimes 1 \wedge \bar{\xi}_1 + \bar{\xi}_1 \otimes \tau_0 \wedge 1 + \bar{\xi}_1 \tau_0 \otimes 1 \wedge 1 \\ \nu(\tau_1 \wedge 1) &= 1 \otimes \tau_1 \wedge 1 + \xi_1 \otimes \tau_0 \wedge 1 + \tau_1 \otimes 1 \wedge 1 \end{aligned}$$

and

$$\nu(1 \wedge y) = 1 \otimes 1 \wedge y + \bar{\tau}_0 \otimes 1 \wedge (\sigma \bar{\tau}_0)^{p-1} - \bar{\tau}_0 \otimes 1 \wedge \bar{\xi}_1 - \bar{\tau}_1 \otimes 1 \wedge 1.$$

Hence the sum

$$\bar{\epsilon}_1 := 1 \wedge y + \tau_0 \wedge (\sigma \bar{\tau}_0)^{p-1} - \tau_0 \wedge \bar{\xi}_1 - \tau_1 \wedge 1 \quad (4.4)$$

is  $A_*$ -comodule primitive. Its image under  $\pi_*$  in  $H_*(V(1) \wedge THH(\mathbb{Z}/p))$  is

$$\epsilon_0 \mu_0^{p-1} - \epsilon_1 = 1 \wedge \bar{\tau}_0 (\sigma \bar{\tau}_0)^{p-1} + \tau_0 \wedge (\sigma \bar{\tau}_0)^{p-1} - 1 \wedge \bar{\tau}_1 - \tau_0 \wedge \bar{\xi}_1 - \tau_1 \wedge 1.$$

Let

$$V(1)_*THH(\ell)/(\lambda_1) = E(\lambda_2) \otimes P(\mu_2)$$

be the quotient algebra of  $V(1)_*THH(\ell)$  by the ideal generated by  $\lambda_1$ .

**Proposition 4.2.** *The classes*

$$1, \epsilon_0, \mu_0, \epsilon_0 \mu_0, \dots, \mu_0^{p-1}, \bar{\epsilon}_1 \in H_*(V(1) \wedge THH(\ell/p))$$

*defined in (4.2), (4.3) and (4.4) have unique lifts with same names in  $V(1)_*THH(\ell/p)$ . The graded  $V(1)_*THH(\ell)$ -module  $V(1)_*THH(\ell/p)$  is a free  $V(1)_*THH(\ell)/(\lambda_1)$ -module generated by these  $2p$  classes:*

$$V(1)_*THH(\ell/p) = V(1)_*THH(\ell)/(\lambda_1) \otimes \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0 \mu_0, \dots, \mu_0^{p-1}, \bar{\epsilon}_1\}.$$

*The map  $\pi_*$  to  $V(1)_*THH(\mathbb{Z}/p)$  takes  $\epsilon_0^\delta \mu_0^i$  in degree  $0 \leq \delta + 2i \leq 2p-2$  to  $\epsilon_0^\delta \mu_0^i$ , and takes  $\bar{\epsilon}_1$  in degree  $(2p-1)$  to  $\epsilon_0 \mu_0^{p-1} - \epsilon_1$ .*

*Proof.* Additively, this follows by another dimension count, and the description of  $\pi_*$  follows from the definition of the classes in question. It remains to prove that the action of  $V(1)_*THH(\ell)$  is as claimed.

The action of  $\mu_2^i$  and  $\lambda_2\mu_2^i$  in  $V(1)_*THH(\ell)$  on the generators

$$1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}, \bar{\epsilon}_1$$

of  $V(1)_*THH(\ell/p)$  is non-trivial for all  $i \geq 0$ , since the corresponding statement holds for the images of these classes in  $H_*(V(1) \wedge THH(\ell))$  and  $H_*(V(1) \wedge THH(\ell/p))$ . This follows from Lemma 3.3 and the definition these classes. It remains to show that  $\lambda_1$  acts trivially on  $V(1)_*THH(\ell/p)$ . For degree reasons, multiplication by  $\lambda_1$  is zero on all classes except possibly  $\mu_2^i$  and  $\lambda_2\mu_2^i$ , for  $i \geq 0$ . Because of the module structure, it suffices to show that  $\lambda_1 = \lambda_1 \cdot 1 = 0$  in  $V(1)_*THH(\ell/p)$ . This follows from the statement that the image of  $\lambda_1$  in  $H_*(V(1) \wedge THH(\ell/p))$  is equal to  $1 \wedge \sigma\bar{\epsilon}_1 = 0$ , as implied by Lemma 3.3.  $\square$

## 5. THE $C_p$ -TATE CONSTRUCTION

For the remainder of this paper, let  $p$  be a prime with  $p \geq 5$ . We briefly recall the terminology on equivariant stable homotopy theory used in the sequel, and refer to [GM95], [HM97, §1], [HM03, §4] and [AR02, §3] for more details. Let  $C_{p^n}$  denote the cyclic group of order  $p^n$ , considered as a closed subgroup of the circle group  $S^1$ , and let  $G = S^1$  or  $C_{p^n}$ . For each spectrum  $X$  with  $S^1$ -action, let  $X_{hG} = EG_+ \wedge_G X$  and  $X^{hG} = F(EG_+, X)^G$  denote its homotopy orbit and homotopy fixed point spectra, as usual. We now write  $X^{tG} = [\widetilde{EG} \wedge F(EG_+, X)]^G$  for the  $G$ -Tate construction on  $X$ , which was denoted  $t_G(X)^G$  in [GM95] and  $\hat{\mathbb{H}}(G, X)$  in [HM97, HM03, AR02].

We denote by  $F$  the Frobenius map  $X^{C_{p^n}} \rightarrow X^{C_{p^{n-1}}}$  given by the inclusion of fixed-point spectra, and by  $V$  the Verschiebung map  $X^{C_{p^{n-1}}} \rightarrow X^{C_{p^n}}$  given by transfer. We shall also consider the homotopy Frobenius, Tate Frobenius and homotopy Verschiebung maps  $F^h: X^{hS^1} \rightarrow X^{hC_{p^n}}$ ,  $F^h: X^{hC_{p^n}} \rightarrow X^{hC_{p^{n-1}}}$ ,  $F^t: X^{tS^1} \rightarrow X^{tC_{p^n}}$  and  $V^h: X^{hC_{p^{n-1}}} \rightarrow X^{hC_{p^n}}$ .

There are conditionally convergent  $G$ -homotopy fixed point and  $G$ -Tate spectral sequences in  $V(1)$ -homotopy for  $X$ , with

$$E_{s,t}^2(G, X) = H_{gp}^{-s}(G; V(1)_t(X)) \implies V(1)_{s+t}(X^{hG})$$

and

$$\hat{E}_{s,t}^2(G, X) = \hat{H}_{gp}^{-s}(G; V(1)_t(X)) \implies V(1)_{s+t}(X^{tG}).$$

Here  $H_{gp}^*(G; V(1)_*(X))$  denotes the group cohomology of  $G$  and  $\hat{H}_{gp}^*(G; V(1)_*(X))$  the Tate cohomology of  $G$ , with coefficients in  $V(1)_*(X)$ . Notice that in our case, with  $X = THH(B)$ , the action of  $G$  on  $V(1)_*(X)$  is trivial, since it is the restriction of an  $S^1$ -action. We write  $H_{gp}^*(C_{p^n}; \mathbb{F}_p) = E(u_n) \otimes P(t)$  and  $\hat{H}_{gp}^*(C_{p^n}; \mathbb{F}_p) = E(u_n) \otimes P(t^{\pm 1})$  with  $u_n$  in degree 1 and  $t$  in degree 2, see for example [Ben98, Prop. 3.5.5] and [HM03, Lem. 4.2.1]. So  $u_n$ ,  $t$  and  $x \in V(1)_t(X)$  have bidegree  $(-1, 0)$ ,  $(-2, 0)$  and  $(0, t)$  in either spectral sequence, respectively. See [HM03, §4.3] for proofs of the multiplicative properties of these spectral sequences. Similarly, we write  $H_{gp}^*(S^1; \mathbb{F}_p) = P(t)$  and  $\hat{H}_{gp}^*(S^1; \mathbb{F}_p) = P(t^{\pm 1})$ . We have morphisms of spectral sequences induced by the homotopy and Tate Frobenii, which on the  $E^2$ -terms map  $t$  to  $t$  and  $u_n$  to zero.

We are principally interested in the case when  $X = THH(B)$ , with the  $S^1$ -action given by the cyclic structure [Lod98, Def. 7.1.9], [HM03, §1.2]. It is a cyclotomic spectrum, in

the sense of [HM97, §1], leading to the commutative diagram

$$\begin{array}{ccccccc}
THH(B)_{hC_{p^n}} & \xrightarrow{N} & THH(B)^{C_{p^n}} & \xrightarrow{R} & THH(B)^{C_{p^{n-1}}} & \longrightarrow & \Sigma THH(B)_{hC_{p^n}} \\
\parallel & & \downarrow \Gamma_n & & \downarrow \hat{\Gamma}_n & & \parallel \\
THH(B)_{hC_{p^n}} & \xrightarrow{N^h} & THH(B)^{hC_{p^n}} & \xrightarrow{R^h} & THH(B)^{tC_{p^n}} & \longrightarrow & \Sigma THH(B)_{hC_{p^n}}
\end{array}$$

of horizontal cofiber sequences. We abbreviate  $\hat{E}^2(G, THH(B))$  to  $\hat{E}^2(G, B)$ , etc. When  $B$  is a commutative  $S$ -algebra, this is a commutative algebra spectral sequence, and when  $B$  is an associative  $A$ -algebra, with  $A$  commutative, then  $\hat{E}^*(G, B)$  is a module spectral sequence over  $\hat{E}^*(G, A)$ . The map  $R^h$  corresponds to the inclusion  $E^2(G, B) \rightarrow \hat{E}^2(G, B)$  from the second quadrant to the upper half-plane, for connective  $B$ .

**Definition 5.1.** We call a homomorphism of graded groups  $k$ -coconnected if it is an isomorphism in all dimensions greater than  $k$  and injective in dimension  $k$ .

In this section we compute  $V(1)_*THH(\ell/p)^{tC_p}$  by means of the  $C_p$ -Tate spectral sequence in  $V(1)$ -homotopy for  $THH(\ell/p)$ . In Propositions 5.7 and 5.8 we show that the comparison map  $\hat{\Gamma}_1: V(1)_*THH(\ell/p) \rightarrow V(1)_*THH(\ell/p)^{tC_p}$  is  $(2p-2)$ -coconnected and can be identified with the algebraic localization homomorphism that inverts  $\mu_2$ .

First we recall the structure of the  $C_p$ -Tate spectral sequence for  $THH(\mathbb{Z}/p)$ , with  $V(0)$ - and  $V(1)$ -coefficients. We have  $V(0)_*THH(\mathbb{Z}/p) = E(\epsilon_0) \otimes P(\mu_0)$ , and (with an obvious notation for the case of  $V(0)$ -homotopy) the  $E^2$ -terms are

$$\begin{aligned}
\hat{E}^2(C_p, \mathbb{Z}/p; V(0)) &= E(u_1) \otimes P(t^{\pm 1}) \otimes E(\epsilon_0) \otimes P(\mu_0) \\
\hat{E}^2(C_p, \mathbb{Z}/p) &= E(u_1) \otimes P(t^{\pm 1}) \otimes E(\epsilon_0, \epsilon_1) \otimes P(\mu_0).
\end{aligned}$$

In each  $G$ -Tate spectral sequence we have a first differential

$$d^2(x) = t \cdot \sigma x,$$

see e.g. [Rog98, §3.3]. We easily deduce  $\sigma\epsilon_0 = \mu_0$  and  $\sigma\epsilon_1 = \mu_0^p$  from (4.1), so

$$\begin{aligned}
\hat{E}^3(C_p, \mathbb{Z}/p; V(0)) &= E(u_1) \otimes P(t^{\pm 1}) \\
\hat{E}^3(C_p, \mathbb{Z}/p) &= E(u_1) \otimes P(t^{\pm 1}) \otimes E(\epsilon_0\mu_0^{p-1} - \epsilon_1).
\end{aligned}$$

Thus the  $V(0)$ -homotopy spectral sequence collapses at  $\hat{E}^3 = \hat{E}^\infty$ . By naturality with respect to the map  $i_1: V(0) \rightarrow V(1)$ , all the classes on the horizontal axis of  $\hat{E}^3(C_p, \mathbb{Z}/p)$  are infinite cycles, so also the latter spectral sequence collapses at  $\hat{E}^3(C_p, \mathbb{Z}/p)$ .

We know from [HM03, Cor. 4.4.2] that the comparison map

$$\hat{\Gamma}_1: V(0)_*THH(\mathbb{Z}/p) \rightarrow V(0)_*THH(\mathbb{Z}/p)^{tC_p}$$

takes  $\epsilon_0^\delta \mu_0^i$  to  $(u_1 t^{-1})^\delta t^{-i}$ , for all  $0 \leq \delta \leq 1$ ,  $i \geq 0$ . In particular, the integral map  $\hat{\Gamma}_1: \pi_*THH(\mathbb{Z}/p) \rightarrow \pi_*THH(\mathbb{Z}/p)^{tC_p}$  is  $(-2)$ -coconnected. From this we can deduce the following behavior of the comparison map  $\hat{\Gamma}_1$  in  $V(1)$ -homotopy.

**Lemma 5.2.** *The map*

$$\hat{\Gamma}_1: V(1)_*THH(\mathbb{Z}/p) \rightarrow V(1)_*THH(\mathbb{Z}/p)^{tC_p}$$

*takes the classes  $\epsilon_0^\delta \mu_0^i$  from  $V(0)_*THH(\mathbb{Z}/p)$ , for  $0 \leq \delta \leq 1$  and  $i \geq 0$ , to classes represented in  $\hat{E}^\infty(C_p, \mathbb{Z}/p)$  by  $(u_1 t^{-1})^\delta t^{-i}$  (on the horizontal axis). Furthermore, it takes the*

class  $\epsilon_0\mu_0^{p-1} - \epsilon_1$  in degree  $(2p-1)$  to a class represented by  $\epsilon_0\mu_0^{p-1} - \epsilon_1$  (on the vertical axis).

*Proof.* The classes  $\epsilon_0\mu_0^i$  are in the image from  $V(0)$ -homotopy, and we recalled above that they are detected by  $(u_1t^{-1})^\delta t^{-i}$  in the  $V(0)$ -homotopy  $C_p$ -Tate spectral sequence for  $THH(\mathbb{Z}/p)$ . By naturality along  $i_1: V(0) \rightarrow V(1)$ , they are detected by the same (nonzero) classes in the  $V(1)$ -homotopy spectral sequence  $\hat{E}^\infty(C_p, \mathbb{Z}/p)$ .

To find the representative for  $\hat{\Gamma}_1(\epsilon_0\mu_0^{p-1} - \epsilon_1)$  in degree  $(2p-1)$ , we appeal to the cyclotomic trace map from algebraic  $K$ -theory, or more precisely, to the commutative diagram

$$\begin{array}{ccccc}
 & & K(B) & & \\
 & \swarrow tr & \downarrow tr_1 & \searrow tr & \\
 THH(B) & \xleftarrow{F} & THH(B)^{C_p} & \xrightarrow{R} & THH(B) \\
 & \nwarrow & \downarrow \Gamma_1 & & \downarrow \hat{\Gamma}_1 \\
 & & THH(B)^{hC_p} & \xrightarrow{R^h} & THH(B)^{tC_p} .
 \end{array} \tag{5.1}$$

The Bökstedt trace map  $tr: K(B) \rightarrow THH(B)$  admits a preferred lift  $tr_n$  through each fixed point spectrum  $THH(B)^{C_{p^n}}$ , which homotopy equalizes the iterated restriction and Frobenius maps  $R^n$  and  $F^n$  to  $THH(B)$ , see [Dun04, §3]. In particular, the  $\sigma$ -operator on  $V(1)_*THH(B)$  is zero on classes in the image of  $tr$ .

In the case  $B = H\mathbb{Z}/p$  we know that  $K(\mathbb{Z}/p)_p \simeq H\mathbb{Z}_p$ , so  $V(1)_*K(\mathbb{Z}/p) = E(\bar{\epsilon}_1)$ , where the  $v_1$ -Bockstein of  $\bar{\epsilon}_1$  is  $-1$ . The Bökstedt trace image  $tr(\bar{\epsilon}_1) \in V(1)_*THH(\mathbb{Z}/p)$  lies in  $\mathbb{F}_p\{\epsilon_1, \epsilon_0\mu_0^{p-1}\}$ , has  $v_1$ -Bockstein  $tr(-1) = -1$  and suspends by  $\sigma$  to 0. Hence

$$tr(\bar{\epsilon}_1) = \epsilon_0\mu_0^{p-1} - \epsilon_1 .$$

As we recalled above, the map  $\hat{\Gamma}_1: \pi_*THH(\mathbb{Z}/p) \rightarrow \pi_*THH(\mathbb{Z}/p)^{tC_p}$  is  $(-2)$ -coconnected, so the corresponding map in  $V(1)$ -homotopy is at least  $(2p-2)$ -coconnected. Thus it takes  $\epsilon_0\mu_0^{p-1} - \epsilon_1$  to a nonzero class in  $V(1)_*THH(\mathbb{Z}/p)^{tC_p}$ , represented somewhere in total degree  $(2p-1)$  of  $\hat{E}^\infty(C_p, \mathbb{Z}/p)$ , in the lower right hand corner of the diagram.

Going down the middle part of the diagram, we reach a class  $(\Gamma_1 \circ tr_1)(\bar{\epsilon}_1)$ , represented in total degree  $(2p-1)$  in the left half-plane  $C_p$ -homotopy fixed point spectral sequence  $E^\infty(C_p, \mathbb{Z}/p)$ . Its image under the edge homomorphism to  $V(1)_*THH(\mathbb{Z}/p)$  equals  $(F \circ tr_1)(\bar{\epsilon}_1) = tr(\bar{\epsilon}_1)$ , hence  $(\Gamma_1 \circ tr_1)(\bar{\epsilon}_1)$  is represented by  $\epsilon_0\mu_0^{p-1} - \epsilon_1$  in  $E_{0,2p-1}^\infty(C_p, \mathbb{Z}/p)$ . Its image under  $R^h$  in the  $C_p$ -Tate spectral sequence is the generator of  $\hat{E}_{0,2p-1}^\infty(C_p, \mathbb{Z}/p) = \mathbb{F}_p\{\epsilon_0\mu_0^{p-1} - \epsilon_1\}$ , hence that generator is the  $E^\infty$ -representative of  $\hat{\Gamma}_1(\epsilon_0\mu_0^{p-1} - \epsilon_1)$ .  $\square$

The  $(2p-2)$ -connected map  $\pi: \ell/p \rightarrow H\mathbb{Z}/p$  induces a  $(2p-1)$ -connected map  $V(1)_*K(\ell/p) \rightarrow V(1)_*K(\mathbb{Z}/p) = E(\bar{\epsilon}_1)$ , by [BM94, Prop. 10.9]. We can lift the algebraic  $K$ -theory class  $\bar{\epsilon}_1$  to  $\ell/p$ . This lift is not unique, but we fix one choice.

**Definition 5.3.** We call

$$\bar{\epsilon}_1^K \in V(1)_{2p-1}K(\ell/p)$$

a chosen class that maps to the generator  $\bar{\epsilon}_1$  in  $V(1)_{2p-1}K(\mathbb{Z}/p) \cong \mathbb{Z}/p$ .

**Lemma 5.4.** *The Bökstedt trace  $tr: V(1)_*K(\ell/p) \rightarrow V(1)_*THH(\ell/p)$  takes  $\bar{\epsilon}_1^K$  to  $\bar{\epsilon}_1$ .*

*Proof.* In the commutative square

$$\begin{array}{ccc} V(1)_*K(\ell/p) & \xrightarrow{tr} & V(1)_*THH(\ell/p) \\ \downarrow \pi_* & & \downarrow \pi_* \\ V(1)_*K(\mathbb{Z}/p) & \xrightarrow{tr} & V(1)_*THH(\mathbb{Z}/p) \end{array}$$

the trace image  $tr(\bar{\epsilon}_1^K)$  in  $V(1)_*THH(\ell/p)$  must map under  $\pi_*$  to  $tr(\bar{\epsilon}_1) = \epsilon_0\mu_0^{p-1} - \epsilon_1$  in  $V(1)_*THH(\mathbb{Z}/p)$ , which by Proposition 4.2 characterizes it as being equal to the class  $\bar{\epsilon}_1$ . Hence  $tr(\bar{\epsilon}_1^K) = \bar{\epsilon}_1$ .  $\square$

Next we turn to the  $C_p$ -Tate spectral sequence  $\hat{E}^*(C_p, \ell/p)$  in  $V(1)$ -homotopy for  $THH(\ell/p)$ . Its  $E^2$ -term is

$$\hat{E}^2(C_p, \ell/p) = E(u_1) \otimes P(t^{\pm 1}) \otimes \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}, \bar{\epsilon}_1\} \otimes E(\lambda_2) \otimes P(\mu_2).$$

We have  $d^2(x) = t \cdot \sigma x$ , where

$$\sigma(\epsilon_0^\delta \mu_0^{i-1}) = \begin{cases} \mu_0^i & \text{for } \delta = 1, 0 < i < p, \\ 0 & \text{otherwise} \end{cases}$$

is readily deduced from (4.1), and  $\sigma(\bar{\epsilon}_1) = 0$  since  $\bar{\epsilon}_1$  is in the image of  $tr$ . Thus

$$\hat{E}^3(C_p, \ell/p) = E(u_1) \otimes P(t^{\pm 1}) \otimes E(\bar{\epsilon}_1) \otimes E(\lambda_2) \otimes P(t\mu_2). \quad (5.2)$$

We prefer to use  $t\mu_2$  rather than  $\mu_2$  as a generator, since it represents multiplication by  $v_2$  (up to a unit factor in  $\mathbb{F}_p$ ) in all module spectral sequences over  $E^*(S^1, \ell)$ , by [AR02, Prop. 4.8].

To proceed, we shall use that  $\hat{E}^*(C_p, \ell/p)$  is a module over the spectral sequence for  $THH(\ell)$ . We therefore recall the structure of the latter spectral sequence, from [AR02, Th. 5.5]. It begins

$$\hat{E}^2(C_p, \ell) = E(u_1) \otimes P(t^{\pm 1}) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu_2).$$

The classes  $\lambda_1$ ,  $\lambda_2$  and  $t\mu_2$  are infinite cycles, and the differentials

$$\begin{aligned} d^{2p}(t^{1-p}) &\doteq t\lambda_1 \\ d^{2p^2}(t^{p-p^2}) &\doteq t^p\lambda_2 \\ d^{2p^2+1}(u_1t^{-p^2}) &\doteq t\mu_2 \end{aligned}$$

leave the terms

$$\begin{aligned} \hat{E}^{2p+1}(C_p, \ell) &= E(u_1, \lambda_1, \lambda_2) \otimes P(t^{\pm p}, t\mu_2) \\ \hat{E}^{2p^2+1}(C_p, \ell) &= E(u_1, \lambda_1, \lambda_2) \otimes P(t^{\pm p^2}, t\mu_2) \\ \hat{E}^{2p^2+2}(C_p, \ell) &= E(\lambda_1, \lambda_2) \otimes P(t^{\pm p^2}) \end{aligned}$$

with  $\hat{E}^{2p^2+2} = \hat{E}^\infty$ , converging to  $V(1)_*THH(\ell)^{tC_p}$ . The comparison map  $\hat{\Gamma}_1$  takes  $\lambda_1$ ,  $\lambda_2$  and  $\mu_2$  to  $\lambda_1$ ,  $\lambda_2$  and  $t^{-p^2}$  (up to a unit factor in  $\mathbb{F}_p$ ), respectively, inducing the algebraic localization map and identification

$$\hat{\Gamma}_1: V(1)_*THH(\ell) \rightarrow V(1)_*THH(\ell)[\mu_2^{-1}] \cong V(1)_*THH(\ell)^{tC_p}.$$

**Lemma 5.5.** *In  $\hat{E}^*(C_p, \ell/p)$ , the class  $u_1 t^{-p}$  supports the nonzero differential*

$$d^{2p^2}(u_1 t^{-p}) \doteq u_1 t^{p^2-p} \lambda_2,$$

*and does not survive to the  $E^\infty$ -term.*

*Proof.* In  $\hat{E}^*(C_p, \ell)$ , there is such a differential. By naturality along  $i: \ell \rightarrow \ell/p$ , it follows that there is also such a differential in  $\hat{E}^*(C_p, \ell/p)$ . It remains to argue that the target class is nonzero at the  $E^{2p^2}$ -term. Considering the  $E^3$ -term in (5.2), the only possible source of a previous differential hitting  $u_1 t^{p^2-p} \lambda_2$  is  $\bar{\epsilon}_1$ , supporting a  $d^{2p^2-2p+1}$ -differential. But  $\bar{\epsilon}_1$  is in an even column and  $u_1 t^{p^2-p} \lambda_2$  is in an odd column. By naturality with respect to the Tate Frobenius map  $F^t: THH(\ell/p)^{tS^1} \rightarrow THH(\ell/p)^{tC_p}$ , any such differential from an even to an odd column must be zero. Indeed, the  $S^1$ -Tate spectral sequence has  $E^2$ -term given by  $P(t^{\pm 1}) \otimes V(1)_* THH(\ell/p)$ , and  $F^t$  induces the injective homomorphism that takes  $\hat{E}^2(S^1, \ell/p)$  isomorphically to the even columns of  $\hat{E}^2(C_p, \ell/p)$ . Since  $\hat{E}^*(S^1, \ell/p)$  is concentrated in even columns, all differentials of odd length are zero. By naturality, classes of  $\hat{E}^r(C_p, \ell/p)$  that lie in the image of  $\hat{E}^r(F^t)$  cannot support a differential of odd length; compare with [AR02, Lemma 5.2]. In the present situation, the  $d^2$ -differential of  $\hat{E}^*(C_p, \ell/p)$  leading to (5.2) is also non-zero in  $\hat{E}^*(S^1, \ell/p)$ , so that we have

$$\hat{E}^3(S^1, \ell/p) = P(t^{\pm 1}) \otimes E(\bar{\epsilon}_1) \otimes E(\lambda_2) \otimes P(t\mu_2).$$

By inspection, if the class  $\bar{\epsilon}_1 \in \hat{E}^2(C_p, \ell/p)$  survives to  $\hat{E}^{2p^2-2p+1}(C_p, \ell/p)$ , then it will lie in the image of  $\hat{E}^{2p^2-2p+1}(F^t)$ .  $\square$

To determine the map  $\hat{\Gamma}_1$  we use naturality with respect to the map  $\pi: \ell/p \rightarrow H\mathbb{Z}/p$ .

**Lemma 5.6.** *The classes  $1, \epsilon_0, \mu_0, \epsilon_0 \mu_0, \dots, \mu_0^{p-1}$  and  $\bar{\epsilon}_1$  in  $V(1)_* THH(\ell/p)$  map under  $\hat{\Gamma}_1$  to classes in  $V(1)_* THH(\ell/p)^{tC_p}$  that are represented in  $\hat{E}^\infty(C_p, \ell/p)$  by the permanent cycles  $(u_1 t^{-1})^\delta t^{-i}$  (on the horizontal axis) in degrees  $\leq (2p-2)$ , and by the permanent cycle  $\bar{\epsilon}_1$  (on the vertical axis) in degree  $(2p-1)$ .*

*Proof.* In the commutative square

$$\begin{array}{ccc} V(1)_* THH(\ell/p) & \xrightarrow{\hat{\Gamma}_1} & V(1)_* THH(\ell/p)^{tC_p} \\ \downarrow \pi_* & & \downarrow \pi_* \\ V(1)_* THH(\mathbb{Z}/p) & \xrightarrow{\hat{\Gamma}_1} & V(1)_* THH(\mathbb{Z}/p)^{tC_p} \end{array}$$

the classes  $\epsilon_0^\delta \mu_0^i$  in the upper left hand corner map to classes in the lower right hand corner that are represented by  $(u_1 t^{-1})^\delta t^{-i}$  in degrees  $\leq (2p-2)$ , and  $\bar{\epsilon}_1$  maps to  $\epsilon_0 \mu_0^{p-1} - \epsilon_1$  in degree  $(2p-1)$ . This follows by combining Proposition 4.2 and Lemma 5.2.

The first  $(2p-1)$  of these are represented in maximal filtration (on the horizontal axis), so their images in the upper right hand corner must be represented by permanent cycles  $(u_1 t^{-1})^\delta t^{-i}$  in the Tate spectral sequence  $\hat{E}^\infty(C_p, \ell/p)$ .

The image of the last class,  $\bar{\epsilon}_1$ , in the upper right hand corner could either be represented by  $\bar{\epsilon}_1$  in bidegree  $(0, 2p-1)$  or by  $u_1 t^{-p}$  in bidegree  $(2p-1, 0)$ . However, the last class supports a differential  $d^{2p^2}(u_1 t^{-p}) \doteq u_1 t^{p^2-p} \lambda_2$ , by Lemma 5.5 above. This only leaves the other possibility, that  $\hat{\Gamma}_1(\bar{\epsilon}_1)$  is represented by  $\bar{\epsilon}_1$  in  $\hat{E}^\infty(C_p, \ell/p)$ .  $\square$



We proceed to determine the differential structure in  $\hat{E}^*(C_p, \ell/p)$ , making use of the permanent cycles identified above.

**Proposition 5.7.** *The  $C_p$ -Tate spectral sequence in  $V(1)$ -homotopy for  $THH(\ell/p)$  has*

$$\hat{E}^3(C_p, \ell/p) = E(u_1, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm 1}, t\mu_2).$$

*It has differentials generated by*

$$d^{2p^2-2p+2}(t^{p-p^2} \cdot t^{-i}\bar{\epsilon}_1) \doteq t\mu_2 \cdot t^{-i}$$

*for  $0 < i < p$ ,  $d^{2p^2}(t^{p-p^2}) \doteq t^p\lambda_2$  and  $d^{2p^2+1}(u_1t^{-p^2}) \doteq t\mu_2$ . The subsequent terms are*

$$\begin{aligned} \hat{E}^{2p^2-2p+3}(C_p, \ell/p) &= E(u_1, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p}) \\ &\quad \oplus E(u_1, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p}, t\mu_2) \\ \hat{E}^{2p^2+1}(C_p, \ell/p) &= E(u_1, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\ &\quad \oplus E(u_1, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^2}, t\mu_2) \\ \hat{E}^{2p^2+2}(C_p, \ell/p) &= E(u_1, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\ &\quad \oplus E(\bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^2}). \end{aligned}$$

*The last term can be rewritten as*

$$\hat{E}^\infty(C_p, \ell/p) = (E(u_1) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \oplus E(\bar{\epsilon}_1)) \otimes E(\lambda_2) \otimes P(t^{\pm p^2}).$$

*Proof.* We have already identified the  $E^2$ - and  $E^3$ -terms above. The  $E^3$ -term (5.2) is generated over  $\hat{E}^3(C_p, \ell)$  by an  $\mathbb{F}_p$ -basis for  $E(\bar{\epsilon}_1)$ , so the next possible differential is induced by  $d^{2p}(t^{1-p}) \doteq t\lambda_1$ . But multiplication by  $\lambda_1$  is trivial in  $V(1)_*THH(\ell/p)$ , by Proposition 4.2, so  $\hat{E}^3(C_p, \ell/p) = \hat{E}^{2p+1}(C_p, \ell/p)$ . This term is generated over  $\hat{E}^{2p+1}(C_p, \ell)$  by  $P_p(t^{-1}) \otimes E(\bar{\epsilon}_1)$ . Here  $1, t^{-1}, \dots, t^{1-p}$  and  $\bar{\epsilon}_1$  are permanent cycles, by Lemma 5.6. Any  $d^r$ -differential before  $d^{2p^2}$  must therefore originate on a class  $t^{-i}\bar{\epsilon}_1$  for  $0 < i < p$ , and be of even length  $r$ , since these classes lie in even columns. For bidegree reasons, the first possibility is  $r = 2p^2 - 2p + 2$ , so  $\hat{E}^3(C_p, \ell/p) = \hat{E}^{2p^2-2p+2}(C_p, \ell/p)$ .

Multiplication by  $v_2$  acts trivially on  $V(1)_*THH(\ell)$  and  $V(1)_*THH(\ell)^{tC_p}$  for degree reasons, and therefore also on  $V(1)_*THH(\ell/p)$  and  $V(1)_*THH(\ell/p)^{tC_p}$  by the module structure. The class  $v_2$  maps to  $t\mu_2$  in the  $S^1$ -Tate spectral sequence for  $\ell$ , as recalled above, so multiplication by  $v_2$  is represented by multiplication by  $t\mu_2$  in the  $C_p$ -Tate spectral sequence for  $\ell/p$ . Applied to the permanent cycles  $(u_1t^{-1})^\delta t^{-i}$  in degrees  $\leq (2p-2)$ , this implies that the products

$$t\mu_2 \cdot (u_1t^{-1})^\delta t^{-i}$$

must be infinite cycles representing zero, i.e., they must be hit by differentials. In the cases  $\delta = 1$ ,  $0 \leq i \leq p-2$ , these classes in odd columns cannot be hit by differentials of odd length, such as  $d^{2p^2+1}$ , so the only possibility is

$$d^{2p^2-2p+2}(t^{p-p^2} \cdot (u_1t^{-1})t^{-i}\bar{\epsilon}_1) \doteq t\mu_2 \cdot (u_1t^{-1})t^{-i}$$

for  $0 \leq i \leq p-2$ . By the module structure (consider multiplication by  $u_1$ ) it follows that

$$d^{2p^2-2p+2}(t^{p-p^2} \cdot t^{-i}\bar{\epsilon}_1) \doteq t\mu_2 \cdot t^{-i}$$

for  $0 < i < p$ . Hence we can compute from (5.2) that

$$\begin{aligned} \hat{E}^{2p^2-2p+3}(C_p, \ell/p) &= E(u_1) \otimes P(t^{\pm p}) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes E(\lambda_2) \\ &\oplus E(u_1) \otimes P(t^{\pm p}) \otimes E(\bar{\epsilon}_1) \otimes E(\lambda_2) \otimes P(t\mu_2). \end{aligned}$$

This is generated over  $\hat{E}^{2p+1}(C_p, \ell)$  by the permanent cycles  $1, t^{-1}, \dots, t^{1-p}$  and  $\bar{\epsilon}_1$ , so the next differential is induced by  $d^{2p^2}(t^{p-p^2}) \doteq t^p \lambda_2$ . This leaves

$$\begin{aligned} \hat{E}^{2p^2+1}(C_p, \ell/p) &= E(u_1) \otimes P(t^{\pm p^2}) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes E(\lambda_2) \\ &\oplus E(u_1) \otimes P(t^{\pm p^2}) \otimes E(\bar{\epsilon}_1) \otimes E(\lambda_2) \otimes P(t\mu_2). \end{aligned}$$

Finally,  $d^{2p^2+1}(u_1 t^{-p^2}) \doteq t\mu_2$  applies, and leaves

$$\begin{aligned} \hat{E}^{2p^2+2}(C_p, \ell/p) &= E(u_1) \otimes P(t^{\pm p^2}) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes E(\lambda_2) \\ &\oplus P(t^{\pm p^2}) \otimes E(\bar{\epsilon}_1) \otimes E(\lambda_2). \end{aligned}$$

For bidegree reasons,  $\hat{E}^{2p^2+2} = \hat{E}^\infty$ . □

**Proposition 5.8.** *The comparison map  $\hat{\Gamma}_1$  takes the classes*

$$\epsilon_0^\delta \mu_0^i, \bar{\epsilon}_1, \lambda_2 \text{ and } \mu_2 \text{ in } V(1)_* THH(\ell/p)$$

*to classes in  $V(1)_* THH(\ell/p)^{tC_p}$  represented by*

$$(u_1 t^{-1})^\delta t^{-i}, \bar{\epsilon}_1, \lambda_2 \text{ and } t^{-p^2} \text{ in } \hat{E}^\infty(C_p, \ell/p),$$

*up to a unit factor in  $\mathbb{F}_p$ , respectively. Thus*

$$V(1)_* THH(\ell/p)^{tC_p} \cong \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0 \mu_0, \dots, \mu_0^{p-1}, \bar{\epsilon}_1\} \otimes E(\lambda_2) \otimes P(\mu_2^{\pm 1})$$

*and  $\hat{\Gamma}_1$  induces an identification  $V(1)_* THH(\ell/p)[\mu_2^{-1}] \cong V(1)_* THH(\ell/p)^{tC_p}$ . In particular,  $\hat{\Gamma}_1$  factors as the algebraic localization map and identification*

$$\hat{\Gamma}_1: V(1)_* THH(\ell/p) \rightarrow V(1)_* THH(\ell/p)[\mu_2^{-1}] \cong V(1)_* THH(\ell/p)^{tC_p},$$

*and is  $(2p-2)$ -coconnected.*

*Proof.* The image under  $\hat{\Gamma}_1$  of the classes  $1, \epsilon_0, \mu_0, \epsilon_0 \mu_0, \dots, \mu_0^{p-1}$  and  $\bar{\epsilon}_1$  was given in Lemma 5.6, and the action on the classes  $\lambda_2$  and  $\mu_2$  is given in the proof of [AR02, Th. 5.5]. The structure of  $V(1)_* THH(\ell/p)^{tC_p}$  is then immediate from the  $E^\infty$ -term in Proposition 5.7. The top class not in the image of  $\hat{\Gamma}_1$  is  $\bar{\epsilon}_1 \lambda_2 \mu_2^{-1}$ , in degree  $(2p-2)$ . □

Recall that

$$\begin{aligned} TF(B; p) &= \operatorname{holim}_{n, F} THH(B)^{C_{p^n}} \\ TR(B; p) &= \operatorname{holim}_{n, R} THH(B)^{C_{p^n}} \end{aligned}$$

are defined as the homotopy limits over the Frobenius and the restriction maps

$$F, R: THH(B)^{C_{p^n}} \rightarrow THH(B)^{C_{p^{n-1}}},$$

respectively.

**Corollary 5.9.** *The comparison maps*

$$\begin{aligned}\Gamma_n &: THH(\ell/p)^{C_{p^n}} \rightarrow THH(\ell/p)^{hC_{p^n}} \\ \hat{\Gamma}_n &: THH(\ell/p)^{C_{p^{n-1}}} \rightarrow THH(\ell/p)^{tC_{p^n}}\end{aligned}$$

for  $n \geq 1$ , and

$$\begin{aligned}\Gamma &: TF(\ell/p; p) \rightarrow THH(\ell/p)^{hS^1} \\ \hat{\Gamma} &: TF(\ell/p; p) \rightarrow THH(\ell/p)^{tS^1}\end{aligned}$$

all induce  $(2p-2)$ -coconnected homomorphisms on  $V(1)$ -homotopy.

*Proof.* This follows from a theorem of Tsalidis [Tsa98, Th. 2.4] and Proposition 5.8 above, just like in [AR02, Th. 5.7]. See also [BBLNR, Ex. 10.2]  $\square$

## 6. HIGHER FIXED POINTS

Let  $n \geq 1$ . Write  $v_p(i)$  for the  $p$ -adic valuation of  $i$ . Define a numerical function  $\rho(-)$  by

$$\begin{aligned}\rho(2k-1) &= (p^{2k+1} + 1)/(p+1) = p^{2k} - p^{2k-1} + \cdots - p + 1 \\ \rho(2k) &= (p^{2k+2} - p^2)/(p^2 - 1) = p^{2k} + p^{2k-2} + \cdots + p^2\end{aligned}$$

for  $k \geq 0$ , so  $\rho(-1) = 1$  and  $\rho(0) = 0$ . For even arguments,  $\rho(2k) = r(2k)$  as defined in [AR02, Def. 2.5].

In all of the following spectral sequences we know that  $\lambda_2$ ,  $t\mu_2$  and  $\bar{\epsilon}_1$  are infinite cycles. For  $\lambda_2$  and  $\bar{\epsilon}_1$  this follows from the  $C_{p^n}$ -fixed point analogue of diagram (5.1), by [AR02, Prop. 2.8] and Lemma 5.4. For  $t\mu_2$  it follows from [AR02, Prop. 4.8], by naturality.

**Theorem 6.1.** *The  $C_{p^n}$ -Tate spectral sequence in  $V(1)$ -homotopy for  $THH(\ell/p)$  begins*

$$\hat{E}^2(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}, \bar{\epsilon}_1\} \otimes P(t^{\pm 1}, \mu_2)$$

and converges to  $V(1)_*THH(\ell/p)^{tC_{p^n}}$ . It is a module spectral sequence over the algebra spectral sequence  $\hat{E}^*(C_{p^n}, \ell)$  converging to  $V(1)_*THH(\ell)^{tC_{p^n}}$ .

There is an initial  $d^2$ -differential generated by

$$d^2(\epsilon_0\mu_0^{i-1}) = t\mu_0^i$$

for  $0 < i < p$ . Next, there are  $2n$  families of even length differentials generated by

$$d^{2\rho(2k-1)}(t^{p^{2k-1}-p^{2k}+i} \cdot \bar{\epsilon}_1) \doteq (t\mu_2)^{\rho(2k-3)} \cdot t^i$$

for  $v_p(i) = 2k-2$ , for each  $k = 1, \dots, n$ , and

$$d^{2\rho(2k)}(t^{p^{2k-1}-p^{2k}}) \doteq \lambda_2 \cdot t^{p^{2k-1}} \cdot (t\mu_2)^{\rho(2k-2)}$$

for each  $k = 1, \dots, n$ . Finally, there is a differential of odd length generated by

$$d^{2\rho(2n)+1}(u_n \cdot t^{-p^{2n}}) \doteq (t\mu_2)^{\rho(2n-2)+1}.$$

We shall prove Theorem 6.1 by induction on  $n$ . The base case  $n = 1$  was covered by Proposition 5.7. We can therefore assume that Theorem 6.1 holds for some fixed  $n \geq 1$ , and must prove the corresponding statement for  $n+1$ . First we make the following deduction.

**Corollary 6.2.** *The initial differential in the  $C_{p^n}$ -Tate spectral sequence in  $V(1)$ -homotopy for  $THH(\ell/p)$  leaves*

$$\hat{E}^3(C_{p^n}, \ell/p) = E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm 1}, t\mu_2).$$

*The next  $2n$  families of differentials leave the intermediate terms*

$$\begin{aligned} \hat{E}^{2\rho(1)+1}(C_{p^n}, \ell/p) &= E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p}) \\ &\oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p}, t\mu_2) \end{aligned}$$

(for  $m = 1$ ),

$$\begin{aligned} \hat{E}^{2\rho(2m-1)+1}(C_{p^n}, \ell/p) &= E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\ &\oplus \bigoplus_{k=2}^m E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 2\} \otimes P_{\rho(2k-3)}(t\mu_2) \\ &\oplus \bigoplus_{k=2}^{m-1} E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \lambda_2 \mid j \in \mathbb{Z}, v_p(j) = 2k - 1\} \otimes P_{\rho(2k-2)}(t\mu_2) \\ &\oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m-1}}, t\mu_2) \end{aligned}$$

for  $m = 2, \dots, n$ , and

$$\begin{aligned} \hat{E}^{2\rho(2m)+1}(C_{p^n}, \ell/p) &= E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\ &\oplus \bigoplus_{k=2}^m E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 2\} \otimes P_{\rho(2k-3)}(t\mu_2) \\ &\oplus \bigoplus_{k=2}^m E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \lambda_2 \mid j \in \mathbb{Z}, v_p(j) = 2k - 1\} \otimes P_{\rho(2k-2)}(t\mu_2) \\ &\oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m}}, t\mu_2) \end{aligned}$$

for  $m = 1, \dots, n$ . The final differential leaves the  $E^{2\rho(2n)+2} = E^\infty$ -term, equal to

$$\begin{aligned} \hat{E}^\infty(C_{p^n}, \ell/p) &= E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\ &\oplus \bigoplus_{k=2}^n E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 2\} \otimes P_{\rho(2k-3)}(t\mu_2) \\ &\oplus \bigoplus_{k=2}^n E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \lambda_2 \mid j \in \mathbb{Z}, v_p(j) = 2k - 1\} \otimes P_{\rho(2k-2)}(t\mu_2) \\ &\oplus E(\bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2n}}) \otimes P_{\rho(2n-2)+1}(t\mu_2). \end{aligned}$$

*Proof.* The statements about the  $E^3$ -,  $E^{2\rho(1)+1}$ - and  $E^{2\rho(2)+1}$ -terms are clear from Proposition 5.7. For each  $m = 2, \dots, n$  we proceed by a secondary induction. The differential

$$d^{2\rho(2m-1)}(t^{p^{2m-1}-p^{2m}+i} \cdot \bar{\epsilon}_1) \doteq (t\mu_2)^{\rho(2m-3)} \cdot t^i$$

for  $v_p(i) = 2m - 2$  is non-trivial only on the summand

$$E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m-2}}, t\mu_2)$$

of the  $E^{2\rho(2m-2)+1} = E^{2\rho(2m-1)}$ -term, with homology

$$E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^j \mid j \in \mathbb{Z}, v_p(j) = 2m - 2\} \otimes P_{\rho(2m-3)}(t\mu_2) \\ \oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m-1}}, t\mu_2).$$

This gives the stated  $E^{2\rho(2m-1)+1}$ -term. Similarly, the differential

$$d^{2\rho(2m)}(t^{p^{2m-1}-p^{2m}}) \doteq \lambda_2 \cdot t^{p^{2m-1}} \cdot (t\mu_2)^{\rho(2m-2)}$$

is non-trivial only on the summand

$$E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m-1}}, t\mu_2)$$

of the  $E^{2\rho(2m-1)+1} = E^{2\rho(2m)}$ -term, with homology

$$E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \lambda_2 \mid j \in \mathbb{Z}, v_p(j) = 2m - 1\} \otimes P_{\rho(2m-2)}(t\mu_2) \\ \oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m}}, t\mu_2).$$

This gives the stated  $E^{2\rho(2m)+1}$ -term. The final differential

$$d^{2\rho(2n)+1}(u_n \cdot t^{-p^{2n}}) \doteq (t\mu_2)^{\rho(2n-2)+1}$$

is non-trivial only on the summand

$$E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2n}}, t\mu_2)$$

of the  $E^{2\rho(2n)+1}$ -term, with homology

$$E(\bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2n}}) \otimes P_{\rho(2n-2)+1}(t\mu_2).$$

This gives the stated  $E^{2\rho(2n)+2}$ -term. At this stage there is no room for any further differentials, since the spectral sequence is concentrated in a narrower horizontal band than the vertical height of the following differentials.  $\square$

Next we compare the  $C_{p^n}$ -Tate spectral sequence with the  $C_{p^n}$ -homotopy fixed point spectral sequence obtained by restricting the  $E^2$ -term to the second quadrant ( $s \leq 0$ ,  $t \geq 0$ ). It is algebraically easier to handle the latter after inverting  $\mu_2$ , which can be interpreted as comparing  $THH(\ell/p)$  with its  $C_p$ -Tate construction.

In general, there is a commutative diagram

$$\begin{array}{ccccc} THH(B)^{C_{p^n}} & \xrightarrow{R} & THH(B)^{C_{p^{n-1}}} & \xrightarrow{\Gamma_{n-1}} & THH(B)^{hC_{p^{n-1}}} \\ \downarrow \Gamma_n & & \downarrow \hat{\Gamma}_n & & \downarrow \hat{\Gamma}_1^{hC_{p^{n-1}}} \\ THH(B)^{hC_{p^n}} & \xrightarrow{R^h} & THH(B)^{tC_{p^n}} & \xrightarrow{G_{n-1}} & (\rho_p^* THH(B)^{tC_p})^{hC_{p^{n-1}}} \end{array} \quad (6.1)$$

Here  $\rho_p^* THH(B)^{tC_p}$  is a notation for the  $S^1$ -spectrum obtained from the  $S^1/C_p$ -spectrum  $THH(B)^{tC_p}$  via the  $p$ -th root isomorphism  $\rho_p: S^1 \rightarrow S^1/C_p$ , and  $G_{n-1}$  is the comparison map from the  $C_{p^{n-1}}$ -fixed points to the  $C_{p^{n-1}}$ -homotopy fixed points of  $\rho_p^* THH(B)^{tC_p}$ , in view of the identification

$$(\rho_p^* THH(B)^{tC_p})^{C_{p^{n-1}}} = THH(B)^{tC_{p^n}}.$$

We are of course considering the case  $B = \ell/p$ . In  $V(1)$ -homotopy all four maps with labels containing  $\Gamma$  are  $(2p-2)$ -coconnected, by Corollary 5.9, so  $G_{n-1}$  is at least  $(2p-1)$ -coconnected. (We shall see in Lemma 6.8 that  $V(1)_* G_{n-1}$  is an isomorphism in all degrees.) By Proposition 5.8 the map  $\hat{\Gamma}_1$  precisely inverts  $\mu_2$ , so the  $E^2$ -term of the  $C_{p^n}$ -homotopy fixed point spectral sequence in  $V(1)$ -homotopy for  $THH(\ell/p)^{tC_p}$  is obtained

by inverting  $\mu_2$  in  $E^2(C_{p^n}, \ell/p)$ . We denote this spectral sequence by  $\mu_2^{-1}E^*(C_{p^n}, \ell/p)$ , even though in later terms only a power of  $\mu_2$  is present.

**Theorem 6.3.** *The  $C_{p^n}$ -homotopy fixed point spectral sequence  $\mu_2^{-1}E^*(C_{p^n}, \ell/p)$  in  $V(1)$ -homotopy for  $THH(\ell/p)^{tC_p}$  begins*

$$\mu_2^{-1}E^2(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}, \bar{\epsilon}_1\} \otimes P(t, \mu_2^{\pm 1})$$

and converges to  $V(1)_*(\rho_p^*THH(\ell/p)^{tC_p})^{hC_{p^n}}$ , which receives a  $(2p-2)$ -coconnected map  $(\hat{\Gamma}_1)^{hC_{p^n}}$  from  $V(1)_*THH(\ell/p)^{hC_{p^n}}$ . There is an initial  $d^2$ -differential generated by

$$d^2(\epsilon_0\mu_0^{i-1}) = t\mu_0^i$$

for  $0 < i < p$ . Next, there are  $2n$  families of even length differentials generated by

$$d^{2\rho(2k-1)}(\mu_2^{p^{2k}-p^{2k-1}+j} \cdot \bar{\epsilon}_1) \doteq (t\mu_2)^{\rho(2k-1)} \cdot \mu_2^j$$

for  $v_p(j) = 2k-2$ , for each  $k = 1, \dots, n$ , and

$$d^{2\rho(2k)}(\mu_2^{p^{2k}-p^{2k-1}}) \doteq \lambda_2 \cdot \mu_2^{-p^{2k-1}} \cdot (t\mu_2)^{\rho(2k)}$$

for each  $k = 1, \dots, n$ . Finally, there is a differential of odd length generated by

$$d^{2\rho(2n)+1}(u_n \cdot \mu_2^{p^{2n}}) \doteq (t\mu_2)^{\rho(2n)+1}.$$

*Proof.* The differential pattern follows from Theorem 6.1 by naturality with respect to the maps of spectral sequences

$$\mu_2^{-1}E^*(C_{p^n}, \ell/p) \xleftarrow{\hat{\Gamma}_1^{hC_{p^n}}} E^*(C_{p^n}, \ell/p) \xrightarrow{R^h} \hat{E}^*(C_{p^n}, \ell/p)$$

induced by  $\hat{\Gamma}_1^{hC_{p^n}}$  and  $R^h$ . The first inverts  $\mu_2$  and the second inverts  $t$ , at the level of  $E^2$ -terms. We are also using that  $t\mu_2$ , the image of  $v_2$ , multiplies as an infinite cycle in all of these spectral sequences.  $\square$

**Corollary 6.4.** *The initial differential in the  $C_{p^n}$ -homotopy fixed point spectral sequence in  $V(1)$ -homotopy for  $THH(\ell/p)^{tC_p}$  leaves*

$$\begin{aligned} \mu_2^{-1}E^3(C_{p^n}, \ell/p) &= E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2^{\pm 1}) \\ &\quad \oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm 1}, t\mu_2). \end{aligned}$$

The next  $2n$  families of differentials leave the intermediate terms

$$\begin{aligned} \mu_2^{-1}E^{2\rho(2m-1)+1}(C_{p^n}, \ell/p) &= E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2^{\pm 1}) \\ &\quad \oplus \bigoplus_{k=1}^m E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2k-2\} \otimes P_{\rho(2k-1)}(t\mu_2) \\ &\quad \oplus \bigoplus_{k=1}^{m-1} E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{\lambda_2\mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2k-1\} \otimes P_{\rho(2k)}(t\mu_2) \\ &\quad \oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m-1}}, t\mu_2) \end{aligned}$$

and

$$\begin{aligned} \mu_2^{-1} E^{2\rho(2m)+1}(C_{p^n}, \ell/p) &= E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2^{\pm 1}) \\ &\oplus \bigoplus_{k=1}^m E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 2\} \otimes P_{\rho(2k-1)}(t\mu_2) \\ &\oplus \bigoplus_{k=1}^m E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{\lambda_2 \mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 1\} \otimes P_{\rho(2k)}(t\mu_2) \\ &\oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m}}, t\mu_2) \end{aligned}$$

for  $m = 1, \dots, n$ . The final differential leaves the  $E^{2\rho(2n)+2} = E^\infty$ -term, equal to

$$\begin{aligned} \mu_2^{-1} E^\infty(C_{p^n}, \ell/p) &= E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2^{\pm 1}) \\ &\oplus \bigoplus_{k=1}^n E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 2\} \otimes P_{\rho(2k-1)}(t\mu_2) \\ &\oplus \bigoplus_{k=1}^n E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{\lambda_2 \mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 1\} \otimes P_{\rho(2k)}(t\mu_2) \\ &\oplus E(\bar{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2n}}) \otimes P_{\rho(2n)+1}(t\mu_2). \end{aligned}$$

*Proof.* The computation of the  $E^3$ -term from the  $E^2$ -term is straightforward. The rest of the proof goes by a secondary induction on  $m = 1, \dots, n$ , very much like the proof of Corollary 6.2. The differential

$$d^{2\rho(2m-1)}(\mu_2^{p^{2m}-p^{2m-1}+j} \cdot \bar{\epsilon}_1) \doteq (t\mu_2)^{\rho(2m-1)} \cdot \mu_2^j$$

for  $v_p(j) = 2m - 2$  is non-trivial only on the summand

$$E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m-2}}, t\mu_2)$$

of the  $E^3 = E^{2\rho(1)}$ -term (for  $m = 1$ ), resp. the  $E^{2\rho(2m-2)+1} = E^{2\rho(2m-1)}$ -term (for  $m = 2, \dots, n$ ). Its homology is

$$\begin{aligned} &E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2m - 2\} \otimes P_{\rho(2m-1)}(t\mu_2) \\ &\oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m-1}}, t\mu_2), \end{aligned}$$

which gives the stated  $E^{2\rho(2m-1)+1}$ -term. The differential

$$d^{2\rho(2m)}(\mu_2^{p^{2m}-p^{2m-1}}) \doteq \lambda_2 \cdot \mu_2^{-p^{2m-1}} \cdot (t\mu_2)^{\rho(2m)}$$

is non-trivial only on the summand

$$E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m-1}}, t\mu_2)$$

of the  $E^{2\rho(2m-1)+1} = E^{2\rho(2m)}$ -term, leaving

$$\begin{aligned} &E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{\lambda_2 \mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2m - 1\} \otimes P_{\rho(2m)}(t\mu_2) \\ &\oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m}}, t\mu_2). \end{aligned}$$

This gives the stated  $E^{2\rho(2m)+1}$ -term. The final differential

$$d^{2\rho(2n)+1}(u_n \cdot \mu_2^{p^{2n}}) \doteq (t\mu_2)^{\rho(2n)+1}$$

is non-trivial only on the summand

$$E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2n}}, t\mu_2)$$

of the  $E^{2\rho(2n)+1}$ -term, with homology

$$E(\bar{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2n}}) \otimes P_{\rho(2n)+1}(t\mu_2).$$

This gives the stated  $E^{2\rho(2n)+2}$ -term. There is no room for any further differentials, since the spectral sequence is concentrated in a narrower vertical band than the horizontal width of the following differentials, so  $E^{2\rho(2n)+2} = E^\infty$ .  $\square$

*Proof of Theorem 6.1.* To make the inductive step to  $C_{p^{n+1}}$ , we use that the first  $d^r$ -differential of odd length in  $\hat{E}^*(C_{p^n}, \ell/p)$  occurs for  $r = r_0 = 2\rho(2n) + 1$ . It follows from [AR02, Lem. 5.2] that the terms  $\hat{E}^r(C_{p^n}, \ell/p)$  and  $\hat{E}^r(C_{p^{n+1}}, \ell/p)$  are isomorphic for  $r \leq 2\rho(2n) + 1$ , via the Frobenius map (taking  $t^i$  to  $t^i$ ) in even columns and the Verschiebung map (taking  $u_n t^i$  to  $u_{n+1} t^i$ ) in odd columns. Furthermore, the differential  $d^{2\rho(2n)+1}$  is zero in the latter spectral sequence. This proves the part of Theorem 6.1 for  $n + 1$  that concerns the differentials leading up to the term

$$\begin{aligned} \hat{E}^{2\rho(2n)+2}(C_{p^{n+1}}, \ell/p) &= E(u_{n+1}, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\ &\oplus \bigoplus_{k=2}^n E(u_{n+1}, \lambda_2) \otimes \mathbb{F}_p\{t^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 2\} \otimes P_{\rho(2k-3)}(t\mu_2) \\ &\oplus \bigoplus_{k=2}^n E(u_{n+1}, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \lambda_2 \mid j \in \mathbb{Z}, v_p(j) = 2k - 1\} \otimes P_{\rho(2k-2)}(t\mu_2) \\ &\oplus E(u_{n+1}, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2n}}, t\mu_2). \end{aligned} \tag{6.2}$$

Next we use the following commutative diagram, where we abbreviate  $THH(B)$  to  $T(B)$  for typographical reasons:

$$\begin{array}{ccccccc} (\rho_p^* T(B)^{tC_p})^{hC_{p^n}} & \xleftarrow{\hat{\Gamma}_1^{hC_{p^n}}} & T(B)^{hC_{p^n}} & \xleftarrow{\Gamma_n} & T(B)^{C_{p^n}} & \xrightarrow{\hat{\Gamma}_{n+1}} & T(B)^{tC_{p^{n+1}}} \\ \downarrow F & & \downarrow F & & \downarrow F & & \downarrow F \\ \rho_p^* T(B)^{tC_p} & \xleftarrow{\hat{\Gamma}_1} & T(B) & \xlongequal{\quad} & T(B) & \xrightarrow{\hat{\Gamma}_1} & \rho_p^* T(B)^{tC_p}. \end{array} \tag{6.3}$$

The horizontal maps all induce  $(2p - 2)$ -coconnected maps in  $V(1)$ -homotopy for  $B = \ell/p$ . Each  $F$  is a Frobenius map, forgetting invariance under a  $C_{p^n}$ -action. Thus the map  $\hat{\Gamma}_{n+1}$  to the right induces an isomorphism of  $E(\lambda_2) \otimes P(v_2)$ -modules in all degrees  $* > (2p - 2)$  from  $V(1)_* THH(\ell/p)^{C_{p^n}}$ , implicitly identified to the left with the abutment of  $\mu_2^{-1} E^*(C_{p^n}, \ell/p)$ , to  $V(1)_* THH(\ell/p)^{tC_{p^{n+1}}}$ , which is the abutment of  $\hat{E}^*(C_{p^{n+1}}, \ell/p)$ . The diagram above ensures that the isomorphism induced by  $\hat{\Gamma}_{n+1}$  is compatible with the one induced by  $\hat{\Gamma}_1$ . By Proposition 5.8 it takes  $\bar{\epsilon}_1$ ,  $\lambda_2$  and  $\mu_2$  to  $\bar{\epsilon}_1$ ,  $\lambda_2$  and  $t^{-p^2}$  up to a unit factor in  $\mathbb{F}_p$ , respectively, and similarly for monomials in these classes.

We focus on the summand

$$E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2n - 2\} \otimes P_{\rho(2n-1)}(t\mu_2)$$

in  $\mu_2^{-1} E^\infty(C_{p^n}, \ell/p)$ , abutting to  $V(1)_* THH(\ell/p)^{C_{p^n}}$  in degrees  $> (2p - 2)$ . In the  $P(v_2)$ -module structure on the abutment, each class  $\mu_2^j$  with  $v_p(j) = 2n - 2$ ,  $j > 0$ , generates a copy of  $P_{\rho(2n-1)}(v_2)$ , since there are no permanent cycles in the same total degree as



$y = (t\mu_2)^{\rho(2n-1)} \cdot \mu_2^j$  that have lower (= more negative) homotopy fixed point filtration. See Lemma 6.5 below for the elementary verification. The  $P(v_2)$ -module isomorphism induced by  $\hat{\Gamma}_{n+1}$  must take this to a copy of  $P_{\rho(2n-1)}(v_2)$  in  $V(1)_*THH(\ell/p)^{tC_{p^{n+1}}}$ , generated by  $t^{-p^2j}$ .

Writing  $i = -p^2j$ , we deduce that for  $v_p(i) = 2n$ ,  $i < 0$ , the infinite cycle  $z = (t\mu_2)^{\rho(2n-1)} \cdot t^i$  must represent zero in the abutment, and must therefore be hit by a differential  $z = d^r(x)$  in the  $C_{p^{n+1}}$ -Tate spectral sequence. Here  $r \geq 2\rho(2n) + 2$ .

Since  $z$  generates a free copy of  $P(t\mu_2)$  in the  $E^{2\rho(2n)+2}$ -term displayed in (6.2), and  $d^r$  is  $P(t\mu_2)$ -linear, the class  $x$  cannot be annihilated by any power of  $t\mu_2$ . This means that  $x$  must be contained in the summand

$$E(u_{n+1}, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2n}}, t\mu_2)$$

of  $\hat{E}^{2\rho(2n)+2}(C_{p^{n+1}}, \ell/p)$ . By an elementary check of bidegrees, see Lemma 6.6 below, the only possibility is that  $x$  has vertical degree  $(2p-1)$ , so that we have differentials

$$d^{2\rho(2n+1)}(t^{p^{2n+1}-p^{2n+2}+i} \cdot \bar{\epsilon}_1) \doteq (t\mu_2)^{\rho(2n-1)} \cdot t^i$$

for all  $i < 0$  with  $v_p(i) = 2n$ . The cases  $i > 0$  follow by the module structure over the  $C_{p^{n+1}}$ -Tate spectral sequence for  $\ell$ . The remaining two differentials,

$$d^{2\rho(2n+2)}(t^{p^{2n+1}-p^{2n+2}}) \doteq \lambda_2 \cdot t^{p^{2n+1}} \cdot (t\mu_2)^{\rho(2n)}$$

and

$$d^{2\rho(2n+2)+1}(u_{n+1} \cdot t^{-p^{2n+2}}) \doteq (t\mu_2)^{\rho(2n)+1}$$

are also present in the  $C_{p^{n+1}}$ -Tate spectral sequence for  $\ell$ , see [AR02, Th. 6.1], hence follow in the present case by the module structure. With this we have established the complete differential pattern asserted by Theorem 6.1.  $\square$

**Lemma 6.5.** *For  $j \in \mathbb{Z}$  with  $v_p(j) = 2n-2$ , where  $n \geq 1$ , there are no classes in  $\mu_2^{-1}E^\infty(C_{p^n}, \ell/p)$  in the same total degree as  $y = (t\mu_2)^{\rho(2n-1)} \cdot \mu_2^j$  that have lower homotopy fixed point filtration.*

*Proof.* The total degree of  $y$  is  $2(p^{2n+2} - p^{2n+1} + p - 1) + 2p^2j \equiv (2p-2) \pmod{2p^{2n}}$ , which is even.

Looking at the formula for  $\mu_2^{-1}E^\infty(C_{p^n}, \ell/p)$  in Corollary 6.4, the classes of lower filtration than  $y$  all lie in the terms

$$E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{\lambda_2\mu_2^i \mid j \in \mathbb{Z}, v_p(i) = 2n-1\} \otimes P_{\rho(2n)}(t\mu_2)$$

and

$$E(\bar{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2n}}) \otimes P_{\rho(2n)+1}(t\mu_2).$$

Those in even total degree and of lower filtration than  $y$  are

$$u_n\lambda_2 \cdot \mu_2^i(t\mu_2)^e, \quad \bar{\epsilon}_1\lambda_2 \cdot \mu_2^i(t\mu_2)^e$$

with  $v_p(i) = 2n-1$ ,  $\rho(2n-1) < e < \rho(2n)$ , and

$$\mu_2^i(t\mu_2)^e, \quad \bar{\epsilon}_1\lambda_2 \cdot \mu_2^i(t\mu_2)^e$$

with  $v_p(i) \geq 2n$ ,  $\rho(2n-1) < e \leq \rho(2n)$ .

The total degree of  $u_n\lambda_2 \cdot \mu_2^i(t\mu_2)^e$  for  $v_p(i) = 2n-1$  is  $(-1) + (2p^2-1) + 2p^2i + (2p^2-2)e \equiv (2p^2-2)(e+1) \pmod{2p^{2n}}$ . For this to agree with the total degree of  $y$ , we must have  $(2p-2) \equiv (2p^2-2)(e+1) \pmod{2p^{2n}}$ , so  $(e+1) \equiv 1/(1+p) \pmod{p^{2n}}$  and  $e \equiv \rho(2n-1) - 1 \pmod{p^{2n}}$ . There is no such  $e$  with  $\rho(2n-1) < e < \rho(2n)$ .

The total degree of  $\bar{\epsilon}_1 \lambda_2 \cdot \mu_2^i (t\mu_2)^e$  for  $v_p(i) = 2n - 1$  is  $(2p - 1) + (2p^2 - 1) + 2p^2 i + (2p^2 - 2)e \equiv 2p + (2p^2 - 2)(e + 1) \pmod{2p^{2n}}$ . To agree with that of  $y$ , we must have  $(2p - 2) \equiv 2p + (2p^2 - 2)(e + 1) \pmod{2p^{2n}}$ , so  $(e + 1) \equiv 1/(1 - p^2) \pmod{p^{2n}}$  and  $e \equiv \rho(2n) \pmod{p^{2n}}$ . There is no such  $e$  with  $\rho(2n - 1) < e < \rho(2n)$ .

The total degree of  $\mu_2^i (t\mu_2)^e$  for  $v_p(i) \geq 2n$  is  $2p^2 i + (2p^2 - 2)e \equiv (2p^2 - 2)e \pmod{2p^{2n}}$ . To agree with that of  $y$ , we must have  $(2p - 2) \equiv (2p^2 - 2)e \pmod{2p^{2n}}$ , so  $e \equiv 1/(1 + p) \equiv \rho(2n - 1) \pmod{p^{2n}}$ . There is no such  $e$  with  $\rho(2n - 1) < e \leq \rho(2n)$ .

The total degree of  $\bar{\epsilon}_1 \lambda_2 \cdot \mu_2^i (t\mu_2)^e$  for  $v_p(i) \geq 2n$  is  $(2p - 1) + (2p^2 - 1) + 2p^2 i + (2p^2 - 2)e$ . To agree modulo  $2p^{2n}$  with that of  $y$ , we must have  $e \equiv \rho(2n) \pmod{p^{2n}}$ . The only such  $e$  with  $\rho(2n - 1) < e \leq \rho(2n)$  is  $e = \rho(2n)$ . But in that case, the total degree of  $\bar{\epsilon}_1 \lambda_2 \cdot \mu_2^i (t\mu_2)^e$  is  $2p + 2p^2 i + (2p^2 - 2)(\rho(2n) + 1) = 2(p^{2n+2} + p - 1) + 2p^2 i$ . To be equal to that of  $y$ , we must have  $2p^2 i + 2p^{2n+1} = 2p^2 j$ , which is impossible for  $v_p(i) \geq 2n$  and  $v_p(j) = 2n - 2$ .  $\square$

**Lemma 6.6.** *For  $v_p(i) = 2n$ ,  $n \geq 1$  and  $z = (t\mu_2)^{\rho(2n-1)} \cdot t^i$ , the only class in*

$$E(u_{n+1}, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2n}}, t\mu_2)$$

*that can support a non-zero differential  $d^r(x) = z$  for  $r \geq 2\rho(2n) + 2$  is (a unit times)*

$$x = t^{p^{2n+1} - p^{2n+2} + i} \cdot \bar{\epsilon}_1.$$

*Proof.* The class  $z$  has total degree  $(2p^2 - 2)\rho(2n - 1) - 2i = 2p^{2n+2} - 2p^{2n+1} + 2p - 2 - 2i \equiv (2p - 2) \pmod{2p^{2n}}$ , which is even, and vertical degree  $2p^2 \rho(2n - 1)$ . Hence  $x$  has odd total degree, and vertical degree at most  $2p^2 \rho(2n - 1) - 2\rho(2n) - 1 = 2p^{2n+2} - 2p^{2n+1} - \dots - 2p^3 - 1$ . This leaves the possibilities

$$u_{n+1} \cdot t^j (t\mu_2)^e, \quad \bar{\epsilon}_1 \cdot t^j (t\mu_2)^e, \quad \lambda_2 \cdot t^j (t\mu_2)^e$$

with  $v_p(j) \geq 2n$  and  $0 \leq e < p^{2n} - p^{2n-1} - \dots - p = \rho(2n - 1) - \rho(2n - 2) - 1$ , and

$$u_{n+1} \bar{\epsilon}_1 \lambda_2 \cdot t^j (t\mu_2)^e$$

with  $v_p(j) \geq 2n$  and  $0 \leq e < p^{2n} - p^{2n-1} - \dots - p - 1 = \rho(2n - 1) - \rho(2n - 2) - 2$ .

The total degree of  $x$  must be one more than the total degree of  $z$ , hence is congruent to  $(2p - 1)$  modulo  $2p^{2n}$ .

The total degree of  $u_{n+1} \cdot t^j (t\mu_2)^e$  is  $-1 - 2j + (2p^2 - 2)e \equiv -1 + (2p^2 - 2)e \pmod{2p^{2n}}$ . To have  $(2p - 1) \equiv -1 + (2p^2 - 2)e \pmod{2p^{2n}}$  we must have  $e \equiv -p/(1 - p^2) \equiv p^{2n} - p^{2n-1} - \dots - p \pmod{p^{2n}}$ , which does not happen for  $e$  in the allowable range.

The total degree of  $\lambda_2 \cdot t^j (t\mu_2)^e$  is  $(2p^2 - 1) - 2j + (2p^2 - 2)e \equiv (2p^2 - 1) + (2p^2 - 2)e \pmod{2p^{2n}}$ . To have  $(2p - 1) \equiv (2p^2 - 1) + (2p^2 - 2)e \pmod{2p^{2n}}$  we must have  $e \equiv -p/(1 + p) \equiv \rho(2n - 1) - 1 \pmod{p^{2n}}$ , which does not happen.

The total degree of  $u_{n+1} \bar{\epsilon}_1 \lambda_2 \cdot t^j (t\mu_2)^e$  is  $-1 + (2p - 1) + (2p^2 - 1) - 2j + (2p^2 - 2)e \equiv (2p - 1) + (2p^2 - 2)(e + 1) \pmod{2p^{2n}}$ . To have  $(2p - 1) \equiv (2p - 1) + (2p^2 - 2)(e + 1) \pmod{2p^{2n}}$  we must have  $(e + 1) \equiv 0 \pmod{p^{2n}}$ , so  $e \equiv p^{2n} - 1 \pmod{p^{2n}}$ , which does not happen.

The total degree of  $\bar{\epsilon}_1 \cdot t^j (t\mu_2)^e$  is  $(2p - 1) - 2j + (2p^2 - 2)e \equiv (2p - 1) + (2p^2 - 2)e \pmod{2p^{2n}}$ . To have  $(2p - 1) \equiv (2p - 1) + (2p^2 - 2)e \pmod{2p^{2n}}$ , we must have  $e \equiv 0 \pmod{p^{2n}}$ , so  $e = 0$  is the only possibility in the allowable range. In that case, a check of total degrees shows that we must have  $j = p^{2n+1} - p^{2n+2} + i$ .  $\square$

**Corollary 6.7.**  $V(1)_* THH(\ell/p)^{C_{p^n}}$  is finite in each degree.

*Proof.* This is clear by inspection of the  $E^\infty$ -term in Corollary 6.2.  $\square$

**Lemma 6.8.** *The map  $G_n$  induces an isomorphism*

$$V(1)_*THH(\ell/p)^{tC_{p^{n+1}}} \xrightarrow{\cong} V(1)_*(\rho_p^*THH(\ell/p)^{tC_p})^{hC_{p^n}}$$

*in all degrees. In the limit over the Frobenius maps  $F$ , there is a map  $G$  inducing an isomorphism*

$$V(1)_*THH(\ell/p)^{tS^1} \xrightarrow{\cong} V(1)_*(\rho_p^*THH(\ell/p)^{tC_p})^{hS^1}. \quad (6.4)$$

*Proof.* As remarked after diagram (6.1),  $G_n$  induces an isomorphism in  $V(1)$ -homotopy above degree  $(2p-2)$ . The permanent cycle  $t^{-p^{2n+2}}$  in  $\hat{E}^\infty(C_{p^{n+1}}, \ell)$  acts invertibly on  $\hat{E}^\infty(C_{p^{n+1}}, \ell/p)$ , and its image  $G_n(t^{-p^{2n+2}}) = \mu_2^{p^{2n}}$  in  $\mu_2^{-1}E^\infty(C_{p^n}, \ell)$  acts invertibly on  $\mu_2^{-1}E^\infty(C_{p^n}, \ell/p)$ . Therefore the module action derived from the  $\ell$ -algebra structure on  $\ell/p$  ensures that  $G_n$  induces isomorphisms in  $V(1)$ -homotopy in all degrees.  $\square$

**Theorem 6.9.** *The isomorphism (6.4) admits the following description at the associated graded level:*

(a) *The associated graded of  $V(1)_*THH(\ell/p)^{tS^1}$  for the  $S^1$ -Tate spectral sequence is*

$$\begin{aligned} \hat{E}^\infty(S^1, \ell/p) &= E(\lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\ &\oplus \bigoplus_{k \geq 2} E(\lambda_2) \otimes \mathbb{F}_p\{t^j \mid j \in \mathbb{Z}, v_p(j) = 2k-2\} \otimes P_{\rho(2k-3)}(t\mu_2) \\ &\oplus \bigoplus_{k \geq 2} E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \lambda_2 \mid j \in \mathbb{Z}, v_p(j) = 2k-1\} \otimes P_{\rho(2k-2)}(t\mu_2) \\ &\oplus E(\bar{\epsilon}_1, \lambda_2) \otimes P(t\mu_2). \end{aligned}$$

(b) *The associated graded of  $V(1)_*THH(\ell/p)^{hS^1}$  for the  $S^1$ -homotopy fixed point spectral sequence maps by a  $(2p-2)$ -coconnected map to*

$$\begin{aligned} \mu_2^{-1}E^\infty(S^1, \ell/p) &= E(\lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2^{\pm 1}) \\ &\oplus \bigoplus_{k \geq 1} E(\lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2k-2\} \otimes P_{\rho(2k-1)}(t\mu_2) \\ &\oplus \bigoplus_{k \geq 1} E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{\lambda_2 \mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2k-1\} \otimes P_{\rho(2k)}(t\mu_2) \\ &\oplus E(\bar{\epsilon}_1, \lambda_2) \otimes P(t\mu_2). \end{aligned}$$

(c) *The isomorphism from (a) to (b) induced by  $G$  takes  $t^{-i}$  to  $\mu_0^i$  for  $0 < i < p$  and  $t^i$  to  $\mu_2^j$  for  $i + p^2j = 0$ , up to a unit factor in  $\mathbb{F}_p$ . Furthermore, it takes multiples by  $\bar{\epsilon}_1$ ,  $\lambda_2$  or  $t\mu_2$  in the source to the same multiples in the target, up to a unit factor in  $\mathbb{F}_p$ .*

*Proof.* Claims (a) and (b) follow by passage to the limit over  $n$  from Corollaries 6.2 and 6.4. Claim (c) follows by passage to the same limit from the formulas for the isomorphism induced by  $\hat{\Gamma}_{n+1}$ , which were given below diagram (6.3).  $\square$

## 7. TOPOLOGICAL CYCLIC HOMOLOGY

By definition, there is a fiber sequence

$$TC(B; p) \xrightarrow{\pi} TF(B; p) \xrightarrow{R-1} TF(B; p) \rightarrow \Sigma TC(B; p)$$

inducing a long exact sequence

$$\dots \xrightarrow{\partial} V(1)_*TC(B; p) \xrightarrow{\pi} V(1)_*TF(B; p) \xrightarrow{R_*^{-1}} V(1)_*TF(B; p) \xrightarrow{\partial} \dots \quad (7.1)$$

in  $V(1)$ -homotopy. By Corollary 5.9, there are  $(2p-2)$ -coconnected maps  $\Gamma$  and  $\hat{\Gamma}$  from  $V(1)_*TF(\ell/p; p)$  to  $V(1)_*THH(\ell/p)^{hS^1}$  and  $V(1)_*THH(\ell/p)^{tS^1}$ , respectively. We model  $V(1)_*TF(\ell/p; p)$  in degrees  $> (2p-2)$  by the map  $\hat{\Gamma}$  to the  $S^1$ -Tate construction. Then, by diagram (6.1),  $R_*$  is modeled in the same range of degrees by the chain of maps below:

$$\begin{array}{ccc} V(1)_*THH(B)^{tS^1} & & V(1)_*THH(B)^{hS^1} \xrightarrow{R_*^h} V(1)_*THH(B)^{tS^1} \\ & \searrow G & \downarrow (\hat{\Gamma}_1)^{hS^1} \\ & & V(1)_*(\rho_p^*THH(B)^{tC_p})^{hS^1}. \end{array}$$

Here  $R^h$  induces a map of spectral sequences

$$E^*(R^h): E^*(S^1, B) \rightarrow \hat{E}^*(S^1, B)$$

(abutting to  $R_*^h$ ), which at the  $E^2$ -term equals the inclusion that algebraically inverts  $t$ . When  $B = \ell/p$ , the left hand map  $G$  is an isomorphism by Lemma 6.8, and the middle (wrong-way) map is  $(2p-2)$ -coconnected.

**Proposition 7.1.** *In degrees  $> (2p-2)$ , the homomorphism*

$$E^\infty(R^h): E^\infty(S^1, \ell/p) \rightarrow \hat{E}^\infty(S^1, \ell/p)$$

*maps*

- (a)  $E(\bar{\epsilon}_1, \lambda_2) \otimes P(t\mu_2)$  *identically to the same expression;*
- (b)  $E(\lambda_2) \otimes \mathbb{F}_p\{\mu_2^{-j}\} \otimes P_{\rho(2k-1)}(t\mu_2)$  *surjectively onto*  $E(\lambda_2) \otimes \mathbb{F}_p\{t^j\} \otimes P_{\rho(2k-3)}(t\mu_2)$  *for each*  $k \geq 2$ ,  $j = dp^{2k-2}$ ,  $0 < d < p^2 - p$  *and*  $p \nmid d$ ;
- (c)  $E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{\lambda_2\mu_2^{-j}\} \otimes P_{\rho(2k)}(t\mu_2)$  *surjectively onto*  $E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j\lambda_2\} \otimes P_{\rho(2k-2)}(t\mu_2)$  *for each*  $k \geq 2$ ,  $j = dp^{2k-1}$  *and*  $0 < d < p$ ;
- (d) *the remaining terms to zero.*

Notice that in the statements (b) and (c) above, we abuse notation and indentify the components of degree  $> 2p-2$  of  $E^\infty(S^1, \ell/p)$  and  $\mu_2^{-1}E^\infty(S^1, \ell/p)$ , using Theorem 6.9(b).

*Proof.* Consider the summands of  $E^\infty(S^1, \ell/p)$  and  $\hat{E}^\infty(S^1, \ell/p)$  given in Theorem 6.9. Clearly, the first term  $E(\lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2)$  goes to zero (these classes are hit by  $d^2$ -differentials), and the last term  $E(\bar{\epsilon}_1, \lambda_2) \otimes P(t\mu_2)$  maps identically to the same term. This proves (a) and part of (d).

For each  $k \geq 1$  and  $j = dp^{2k-2}$  with  $p \nmid d$ , the term  $E(\lambda_2) \otimes \mathbb{F}_p\{\mu_2^{-j}\} \otimes P_{\rho(2k-1)}(t\mu_2)$  maps to the term  $E(\lambda_2) \otimes \mathbb{F}_p\{t^j\} \otimes P_{\rho(2k-3)}(t\mu_2)$ , except that the target is zero for  $k = 1$ . In symbols, the element  $\lambda_2^\delta \mu_2^{-j} (t\mu_2)^i$  maps to the element  $\lambda_2^\delta t^j (t\mu_2)^{i-j}$ . If  $d < 0$ , then the  $t$ -exponent in the target is bounded above by  $dp^{2k-2} + \rho(2k-3) < 0$ , so the target lives in the right half-plane and is essentially not hit by the source, which lives in the left half-plane. If  $d > p^2 - p$ , then the total degree in the source is bounded above by  $(2p^2 - 1) - 2dp^{2k} + \rho(2k-1)(2p^2 - 2) < 2p-2$ , so the source lives in total degree  $< (2p-2)$  and will be disregarded. If  $0 < d < p^2 - p$ , then  $\rho(2k-1) - dp^{2k-2} > \rho(2k-3)$  and  $-dp^{2k-2} < 0$ , so the source surjects onto the target. This proves (b) and part of (d).

Lastly, for each  $k \geq 1$  and  $j = dp^{2k-1}$  with  $p \nmid d$ , the term  $E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{\lambda_2\mu_2^{-j}\} \otimes P_{\rho(2k)}(t\mu_2)$  maps to the term  $E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j\lambda_2\} \otimes P_{\rho(2k-2)}(t\mu_2)$ . The target is zero for  $k = 1$ . If  $d < 0$ ,

then  $dp^{2k-1} + \rho(2k-2) < 0$  so the target lives in the right half-plane. If  $d > p$ , then  $(2p-1) + (2p^2-1) - 2dp^{2k+1} + \rho(2k)(2p^2-2) < 2p-2$ , so the source lives in total degree  $< (2p-2)$ . If  $0 < d < p$ , then  $\rho(2k) - dp^{2k-1} > \rho(2k-2)$  and  $-dp^{2k-1} < 0$ , so the source surjects onto the target. This proves (c) and the remaining part of (d).  $\square$

**Definition 7.2.** Let

$$\begin{aligned} A &= E(\bar{\epsilon}_1, \lambda_2) \otimes P(t\mu_2) \\ B_k &= E(\lambda_2) \otimes \mathbb{F}_p\{t^{dp^{2k-2}} \mid 0 < d < p^2 - p, p \nmid d\} \otimes P_{\rho(2k-3)}(t\mu_2) \\ C_k &= E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^{dp^{2k-1}} \lambda_2 \mid 0 < d < p\} \otimes P_{\rho(2k-2)}(t\mu_2) \end{aligned}$$

for  $k \geq 2$  and let  $D$  be the span of the remaining monomials in  $\hat{E}^\infty(S^1, \ell/p)$ . Let  $B = \bigoplus_{k \geq 2} B_k$  and  $C = \bigoplus_{k \geq 2} C_k$ . Then  $\hat{E}^\infty(S^1, \ell/p) = A \oplus B \oplus C \oplus D$ .

**Proposition 7.3.** *In degrees  $> (2p-2)$ , there are closed subgroups  $\tilde{A} = E(\bar{\epsilon}_1, \lambda_2) \otimes P(v_2)$ ,  $\tilde{B}_k$ ,  $\tilde{C}_k$  and  $\tilde{D}$  in  $V(1)_*TF(\ell/p; p)$ , represented by the subgroups  $A$ ,  $B_k$ ,  $C_k$  and  $D$  of  $\hat{E}^\infty(S^1, \ell/p)$ , respectively, such that the homomorphism  $R_* = V(1)_*R$  induced by the restriction map  $R$*

- (a) *is the identity on  $\tilde{A}$ ;*
- (b) *maps  $\tilde{B}_{k+1}$  surjectively onto  $\tilde{B}_k$  for all  $k \geq 2$ ;*
- (c) *maps  $\tilde{C}_{k+1}$  surjectively onto  $\tilde{C}_k$  for all  $k \geq 2$ ;*
- (d) *is zero on  $\tilde{B}_2$ ,  $\tilde{C}_2$  and  $\tilde{D}$ .*

*In these degrees,  $V(1)_*TF(\ell/p; p) \cong \tilde{A} \oplus \tilde{B} \oplus \tilde{C} \oplus \tilde{D}$ , where  $\tilde{B} = \prod_{k \geq 2} \tilde{B}_k$  and  $\tilde{C} = \prod_{k \geq 2} \tilde{C}_k$ .*

*Proof.* The proof is the same as the proof of [AR02, Thm. 7.7], except that in the present paper we work with the Tate model  $THH(\ell/p)^{tS^1}$  for  $TF(\ell/p; p)$ , in place of the homotopy fixed point model  $THH(\ell/p)^{hS^1}$ . The computations are made in  $V(1)$ -homotopy, and we disregard all classes in total degrees  $\leq (2p-2)$ . For example with this convention we write  $\mu_2^{-1}E^\infty(S^1, \ell/p) \cong E^\infty(S^1, \ell/p)$ , using the same abuse of notation as in Proposition 7.1.

In these terms, the restriction homomorphism  $R_*$  is given at the level of  $E^\infty$ -terms as the composite of the isomorphism

$$G_*: \hat{E}^\infty(S^1, \ell/p) \rightarrow \mu_2^{-1}E^\infty(S^1, \ell/p) \cong E^\infty(S^1, \ell/p)$$

and the map

$$E^\infty(R^h): E^\infty(S^1, \ell/p) \rightarrow \hat{E}^\infty(S^1, \ell/p).$$

As an endomorphism of  $\hat{E}^\infty(S^1, \ell/p)$ , this composite  $E^\infty(R^h)G_*$  is the identity on  $A$ , maps  $B_{k+1}$  onto  $B_k$  and  $C_{k+1}$  onto  $C_k$  for all  $k \geq 2$ , and is zero on  $B_2$ ,  $C_2$  and  $D$ , by Theorem 6.9(c) and Proposition 7.1. The task is to find closed lifts of these groups to  $V(1)_*TF(\ell/p; p)$  such that  $R_*$  has similar properties.

Let  $\tilde{A} = E(\bar{\epsilon}_1, \lambda_2) \otimes P(v_2) \subset V(1)_*TF(\ell/p; p)$  be the (degreewise finite, hence closed) subalgebra generated by the images of the classes  $\bar{\epsilon}_1^K$ ,  $\lambda_2$  and  $v_2$  in  $V(1)_*K(\ell/p)$ . Then  $\tilde{A}$  lifts  $A$  and consists of classes in the image of the trace map from  $V(1)_*K(\ell/p)$ . Hence  $R_*$  is the identity on  $\tilde{A}$ .

We fix  $k \geq 2$  and choose, for all  $n \geq 0$ , a subgroup  $B_k^n \subset B_{k+n}$ , as follows. We take

$$B_k^0 = B_k \cap \ker(E^\infty(R^h)G_*)$$

$$= \begin{cases} B_2 & \text{for } k = 2, \\ E(\lambda_2) \otimes \bigoplus_{0 < d < p^2 - p, p \nmid d} \mathbb{F}_p\{t^{dp^{2k-2}}\} \otimes P_{dp^{2k-4} + \rho(2k-5)}^{\rho(2k-3)-1}(t\mu_2) & \text{for } k \geq 3, \end{cases}$$

where  $P_a^b(t\mu_2) = \mathbb{F}_p\{(t\mu_2)^c \mid a \leq c \leq b\}$ . We proceed by induction on  $n$  for  $n \geq 1$ , choosing a subgroup  $B_k^n$  of  $B_{k+n}$  mapping isomorphically onto  $B_k^{n-1}$  under  $E^\infty(R^h)G_*$  (such a group exists by Theorem 6.9(c) and Proposition 7.1(b)). We then have

$$B_k = \bigoplus_{n=0}^{k-2} B_{k-n}^n.$$

By the argument given on top of page 31 of [AR02], we can choose a lift  $\tilde{B}_k^0$  of  $B_k^0$  with

$$\tilde{B}_k^0 \subset \text{im}(R_*) \cap \ker(R_*)$$

in  $V(1)_*TF(\ell/p; p)$ . By induction on  $n \geq 1$ , we choose a lift  $\tilde{B}_k^n \subset \text{im}(R_*)$  of  $B_k^n$  mapping isomorphically onto  $\tilde{B}_k^{n-1}$  under  $R_*$ . Such a choice is possible since the image of  $R_*$  on  $V(1)_*TF(\ell/p; p)$  equals the image of its restriction to  $\text{im}(R_*)$ , see [AR02, p. 30]. Now

$$\tilde{B}_k = \bigoplus_{n=0}^{k-2} \tilde{B}_{k-n}^n$$

is a (degreewise finite, hence closed) lift of  $B_k$  with  $R_*(\tilde{B}_2) = 0$  and  $R_*(\tilde{B}_k) = \tilde{B}_{k-1}$  for  $k \geq 3$ .

To construct  $\tilde{C}_k$  we proceed as for  $\tilde{B}_k$  above, starting with  $C_2^0 = C_2$  and

$$\begin{aligned} C_k^0 &= C_k \cap \ker(E^\infty(R^h)G_*) \\ &= E(\bar{\epsilon}_1) \otimes \bigoplus_{0 < d < p} \mathbb{F}_p\{t^{dp^{2k-1}}\lambda_2\} \otimes P_{dp^{2k-3} + \rho(2k-4)}^{\rho(2k-2)-1}(t\mu_2) \end{aligned}$$

for  $k \geq 3$ , and using Theorem 6.9(c) and Proposition 7.1(c) to choose  $C_k^n$  for  $n \geq 1$ .

It remains to construct  $\tilde{D}$ . By Proposition 7.1(d), the isomorphism  $G_*$  maps  $D$  into  $\ker(E^\infty(R^h))$ . By [AR02, Lem. 7.3] the representatives in  $E^\infty(S^1, \ell/p)$  of the kernel of  $R_*^h$  equal the kernel of  $E^\infty(R^h)$ . It follows that the representatives in  $\hat{E}^\infty(S^1, \ell/p)$  of the kernel of  $R_*$  are mapped isomorphically by  $G_*$  to  $\ker(E^\infty(R^h))$ . Hence we can pick a vector space basis for  $D$ , choose a representative in  $\ker(R_*) \subset V(1)_*TF(\ell/p; p)$  of each basis element, and let  $\tilde{D} \subset V(1)_*TF(\ell/p; p)$  be the closure of the vector space spanned by these chosen representatives. This closure is contained in  $\ker(R_*)$  since  $R_*$  is continuous. Hence  $R_*$  is zero on  $\tilde{D}$ .  $\square$

**Proposition 7.4.** *In degrees  $> (2p - 2)$  there are isomorphisms*

$$\begin{aligned} \ker(R_* - 1) &\cong \tilde{A} \oplus \lim_k \tilde{B}_k \oplus \lim_k \tilde{C}_k \\ &\cong E(\bar{\epsilon}_1, \lambda_2) \otimes P(v_2) \\ &\quad \oplus E(\lambda_2) \otimes \mathbb{F}_p\{t^d \mid 0 < d < p^2 - p, p \nmid d\} \otimes P(v_2) \\ &\quad \oplus E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0 < d < p\} \otimes P(v_2) \end{aligned}$$

and  $\text{cok}(R_* - 1) \cong \tilde{A} = E(\bar{\epsilon}_1, \lambda_2) \otimes P(v_2)$ . Hence there is an isomorphism

$$\begin{aligned} V(1)_*TC(\ell/p; p) &\cong E(\partial, \bar{\epsilon}_1, \lambda_2) \otimes P(v_2) \\ &\oplus E(\lambda_2) \otimes \mathbb{F}_p\{t^d \mid 0 < d < p^2 - p, p \nmid d\} \otimes P(v_2) \\ &\oplus E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0 < d < p\} \otimes P(v_2) \end{aligned}$$

in these degrees, where  $\partial$  has degree  $-1$  and represents the image of  $1$  under the connecting map  $\partial$  in (7.1).

*Proof.* By Proposition 7.3, the homomorphism  $R_* - 1$  is zero on  $\tilde{A}$  and an isomorphism on  $\tilde{D}$ . Furthermore, there is an exact sequence

$$0 \rightarrow \lim_k \tilde{B}_k \rightarrow \prod_{k \geq 2} \tilde{B}_k \xrightarrow{R_* - 1} \prod_{k \geq 2} \tilde{B}_k \rightarrow \lim_k^1 \tilde{B}_k \rightarrow 0$$

and similarly for the  $C$ 's. The derived limit on the right vanishes since each  $\tilde{B}_{k+1}$  surjects onto  $\tilde{B}_k$ .

Multiplication by  $t\mu_2$  in each  $B_k$  is realized by multiplication by  $v_2$  in  $\tilde{B}_k$ . Each  $\tilde{B}_k$  is a sum of  $2(p-1)^2$  cyclic  $P(v_2)$ -modules, and since  $\rho(2k-3)$  grows to infinity with  $k$  their limit is a free  $P(v_2)$ -module of the same rank, with the indicated generators  $t^d$  and  $t^d\lambda_2$  for  $0 < d < p^2 - p, p \nmid d$ . The argument for the  $C$ 's is practically the same.

The long exact sequence (7.1) yields the short exact sequence

$$0 \rightarrow \Sigma^{-1} \text{cok}(R_* - 1) \xrightarrow{\partial} V(1)_*TC(\ell/p; p) \xrightarrow{\pi} \ker(R_* - 1) \rightarrow 0,$$

from which the formula for the middle term follows.  $\square$

*Remark 7.5.* A more obvious set of  $E(\lambda_2) \otimes P(v_2)$ -module generators for  $\lim_k \tilde{B}_k$  would be the classes  $t^{dp^2}$  in  $B_2 \cong \tilde{B}_2$ , for  $0 < d < p^2 - p, p \nmid d$ . We have a commutative diagram

$$\begin{array}{ccc} TF(\ell/p; p) & \xrightarrow{\hat{\Gamma}} & THH(\ell/p)^{tS^1} \\ \downarrow & & \downarrow FG \\ THH(\ell/p)^{C_p} & \xrightarrow{G_1\hat{\Gamma}_2} & (\rho_p^*THH(\ell/p)^{tC_p})^{hC_p}. \end{array}$$

Under the left-hand canonical map  $TF(\ell/p; p) \rightarrow THH(\ell/p)^{C_p}$ , modeled here by

$$FG: THH(\ell/p)^{tS^1} \rightarrow (\rho_p^*THH(\ell/p)^{tC_p})^{hS^1} \rightarrow (\rho_p^*THH(\ell/p)^{tC_p})^{hC_p},$$

the class  $t^{dp^2}$  maps to the class  $\mu_2^{-d}$ . Since we are only concerned with degrees  $> (2p-2)$  we may equally well use its  $v_2$ -power multiple  $(t\mu_2)^d \cdot \mu_2^{-d} = t^d$  as generator, with the advantage that it is in the image of the localization map

$$THH(\ell/p)^{hC_p} \rightarrow (\rho_p^*THH(\ell/p)^{tC_p})^{hC_p}.$$

Hence the class denoted  $t^d$  in  $\lim_k \tilde{B}_k$  is chosen so as to map under  $TF(\ell/p; p) \rightarrow THH(\ell/p)^{hC_p}$  to  $t^d$  in  $E^\infty(C_p, \ell/p)$ . Similarly, the class denoted  $t^{dp}\lambda_2$  in  $\lim_k \tilde{C}_k$  is chosen so as to map to  $t^{dp}\lambda_2$  in  $E^\infty(C_p, \ell/p)$ .

The map  $\pi: \ell/p \rightarrow \mathbb{Z}/p$  is  $(2p-2)$ -connected, hence induces  $(2p-1)$ -connected maps  $\pi_*: K(\ell/p) \rightarrow K(\mathbb{Z}/p)$  and  $\pi_*: V(1)_*TC(\ell/p; p) \rightarrow V(1)_*TC(\mathbb{Z}/p; p)$ , by [BM94, Prop. 10.9] and [Dun97, p. 224]. Here  $TC(\mathbb{Z}/p; p) \simeq H\mathbb{Z}_p \vee \Sigma^{-1}H\mathbb{Z}_p$  and we have an

isomorphism  $V(1)_*TC(\mathbb{Z}/p; p) \cong E(\partial, \bar{\epsilon}_1)$ , so we can recover  $V(1)_*TC(\ell/p; p)$  in degrees  $\leq (2p - 2)$  from this map.

**Theorem 7.6.** *There is an isomorphism of  $P(v_2)$ -modules*

$$\begin{aligned} V(1)_*TC(\ell/p; p) &\cong P(v_2) \otimes E(\partial, \bar{\epsilon}_1, \lambda_2) \\ &\oplus P(v_2) \otimes E(\mathrm{dlog} v_1) \otimes \mathbb{F}_p\{t^d v_2 \mid 0 < d < p^2 - p, p \nmid d\} \\ &\oplus P(v_2) \otimes E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^{dp} \lambda_2 \mid 0 < d < p\} \end{aligned}$$

where  $\mathrm{dlog} v_1 \cdot t^d v_2 = t^d \lambda_2$ . The degrees are  $|\partial| = -1$ ,  $|\bar{\epsilon}_1| = |\lambda_1| = 2p - 1$ ,  $|\lambda_2| = 2p^2 - 1$ ,  $|v_2| = 2p^2 - 2$ ,  $|t| = -2$  and  $|\mathrm{dlog} v_1| = 1$ .

The notation  $\mathrm{dlog} v_1$  for the multiplier  $v_2^{-1} \lambda_2$  is suggested by the relation  $v_1 \cdot \mathrm{dlog} p = \lambda_1$  in  $V(0)_*TC(\mathbb{Z}_{(p)}|\mathbb{Q}; p)$ .

*Proof.* Only the additive generators  $t^d$  for  $0 < d < p^2 - p$ ,  $p \nmid d$  from Proposition 7.4 do not appear in  $V(1)_*TC(\ell/p; p)$ , but their multiples by  $\lambda_2$  and positive powers of  $v_2$  do. This leads to the given formula, where  $\mathrm{dlog} v_1 \cdot t^d v_2$  must be read as  $t^d \lambda_2$ .  $\square$

By [HM97, Th. C] the cyclotomic trace map of [BHM93] induces cofiber sequences

$$K(B_p)_p \xrightarrow{\mathrm{trc}} TC(B; p)_p \xrightarrow{g} \Sigma^{-1} H\mathbb{Z}_p \rightarrow \Sigma K(B_p)_p$$

for each connective  $S$ -algebra  $B$  with  $\pi_0(B_p) = \mathbb{Z}_p$  or  $\mathbb{Z}/p$ , and thus long exact sequences

$$\cdots \rightarrow V(1)_*K(B_p) \xrightarrow{\mathrm{trc}} V(1)_*TC(B; p) \xrightarrow{g} \Sigma^{-1} E(\bar{\epsilon}_1) \rightarrow \cdots$$

This uses the identifications  $W(\mathbb{Z}_p)_F \cong W(\mathbb{Z}/p)_F \cong \mathbb{Z}_p$  of Frobenius coinvariants of rings of Witt vectors, and applies in particular for  $B = H\mathbb{Z}_{(p)}$ ,  $H\mathbb{Z}/p$ ,  $\ell$  and  $\ell/p$ .

**Theorem 7.7.** *There is an isomorphism of  $P(v_2)$ -modules*

$$\begin{aligned} V(1)_*K(\ell/p) &\cong P(v_2) \otimes E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{1, \partial \lambda_2, \lambda_2, \partial v_2\} \\ &\oplus P(v_2) \otimes E(\mathrm{dlog} v_1) \otimes \mathbb{F}_p\{t^d v_2 \mid 0 < d < p^2 - p, p \nmid d\} \\ &\oplus P(v_2) \otimes E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^{dp} \lambda_2 \mid 0 < d < p\}. \end{aligned}$$

*This is a free  $P(v_2)$ -module of rank  $(2p^2 - 2p + 8)$  and of zero Euler characteristic.*

*Proof.* In the case  $B = \mathbb{Z}/p$ ,  $K(\mathbb{Z}/p)_p \simeq H\mathbb{Z}_p$  and the map  $g$  is split surjective up to homotopy. So the induced homomorphism to  $V(1)_*\Sigma^{-1}H\mathbb{Z}_p = \Sigma^{-1}E(\bar{\epsilon}_1)$  is surjective. Since  $\pi: \ell/p \rightarrow \mathbb{Z}/p$  induces a  $(2p - 1)$ -connected map in topological cyclic homology, and  $\Sigma^{-1}E(\bar{\epsilon}_1)$  is concentrated in degrees  $\leq (2p - 2)$ , it follows by naturality that also in the case  $B = \ell/p$  the map  $g$  induces a surjection in  $V(1)$ -homotopy. The kernel of the surjection  $P(v_2) \otimes E(\partial, \bar{\epsilon}_1, \lambda_2) \rightarrow \Sigma^{-1}E(\bar{\epsilon}_1)$  gives the first row in the asserted formula.  $\square$

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